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A FIRST COURSE IN THE CALCULUS

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PREFACE

A MODERN mathematical wit has said that the Calculus is that branch of mathematics which schoolboys understand but senior wranglers fail to comprehend. There is no gainsaying the truth of this aphorism. So long as the graphical treatment and practical applications of the Calculus are kept in view, the subject is an extremely easy and attractive one. Boys can be taught the subject early in their mathematical career, and there is no part of their mathematical training that they enjoy better or which opens up to them wider fields of useful exploration. On the other hand, as soon as the philosophical aspect of the subject is broached, a very different state of affairs supervenes. Experience in the class-room has shown that only a very small proportion of the pupils with whom the teacher has to deal, can in the remotest degree comprehend the rigorous discussion of the Calculus as based on modern arithmetical logic. It has been well said in a recent article on mathematical teaching that the rigorous study of the Calculus based on the strict and logical concept of "number" is a *terminus ad quem* and not an *initium a quo*. If, on the other hand, the elements of the Calculus be based on what a writer has lately called "geometrical imagery," impossible efforts of the immature mind are not required, and the young student finds the study of the Calculus not too hard and well worth his efforts.

In the present text-book no attempt whatever has been made to base the subject on the modern concept of Number and the Continuum. It is assumed that the student has drawn a considerable number of graphs dealing with statistical data, physical phenomena, etc., and that he has a clear-cut notion of what these curves are like as they present themselves in practical work. It is also found from experience in teaching that the young student has no difficulty in grasping the difference between a chord joining two points on these graphs and the tangent at a point. Upon these two graphical principles the treatment in the present book is based. The student is asked to solve a number of practical problems *approximately* by means of squared paper, and the Calculus is thereafter introduced as the means of calculating *accurately* what before was obtained only approximately. There is no attempt whatever to discuss the philosophy of "limits" and such like.

The appeal throughout the course is made to what a celebrated Cambridge mathematician has called the method of "pictures and plausibility." It is hoped, however, that the student will have nothing to unlearn, should he elect afterwards to proceed to a rigorous course of Modern Analysis. He sees throughout the book what is definitely proved and what is merely allowed to rest on a basis of geometric imagination.

In the first ten chapters Differentiation and Integration of powers of x only are dealt with, and the student is shown how to apply the Calculus to easy questions in the Geometry of simple curves, in Mechanics, in Physics, and in Engineering, and to the calculation of Moments of Inertia and the determination of the positions of Centres of Gravity and Centres of Pressure. The Differentiation and Integration of Trigonometrical Functions is next considered, with practical applications. Then follows the Differentiation of Products and Quotients, the use of Parameters, and the Differentiation of the Inverse Trigonometrical Functions. Chapters are given explaining the application of the Calculus to Approximations and Errors and Miscellaneous Methods of Integration, and showing how the Integral Calculus may be applied to evaluate the Lengths of Curves and Areas of Surfaces. The succeeding chapter treats of the Differentiation and Integration of Logarithms, with numerous applications to Physics and Engineering. The last chapters deal with the use of Taylor and Maclaurin's Theorems for Expansions and Calculations, with Power Series and with Differential Equations.

In the presentation of logarithmic differentiation the relation between the gradient of the tangent at any point x on the curve $y = \log_{10} x$ and that of the tangent at the point where the curve crosses the axis of x (*i.e.* where $x = 1$) is established, and then the whole difficulty is seen by the student to rest on the calculation of the curve's gradient where $x=1$. This can be done to as great a degree of accuracy as required by the simple process of repeated extraction of square roots. This is undoubtedly the only method by which the immature mathematician can face the problem. The numerical calculation involved in prolonged square-root extraction may be rather tedious, but the student never loses his intellectual foothold and can see his way right through. e is introduced as that logarithmic base for which the differential of the logarithm is $\frac{1}{x}$, and thus the awkward numerical factor in $\frac{d\log_{10}x}{dx}=\frac{0{\cdot}4343}{x}$ is avoided. Thus e can be computed from the table of logarithms to the base 10,—which can itself be regarded as having been computed from repeated root-extractions and proportional interpolation, as recommended by Professor John Perry in his *Practical Mathematics*, page 86. The usual rigorous process of finding the differential of the logarithm, involving as it does elaborate theorems on Convergence of Series and the very difficult limits $\mathrm{Lt}_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, $\mathrm{Lt}_{h\to 0}\frac{e^h-1}{h}$, etc., has always to be "taken as read" when teaching young students. On the other hand, they see no reason for the results obtained, if they are merely quoted. The above method has stood the test of practical teaching experience. Similarly, Taylor's and Maclaurin's Theorems are not presented as abstract theorems but as a means of interpolating numerical values between functional values easily computed in the case of $\sin x$, $\cos x$, $\log x$, etc. Graphical methods are freely employed. Thus it is hoped that a clear idea of the practical use of these theorems and the cases when they cannot be applied will be obtained.

We have drawn freely for examples from the Qualifying Papers to the Mechanical Science Tripos, and also from the Examination Papers set by the Board of Education, the Army Examiners, the Oxford and Cambridge Joint Board, the various Colleges of the University of Cambridge, etc.

We have pleasure, therefore, in recording our obligations for the use of these various sources of questions to the respective authorities who control the several examinations.

It remains for us to thank once more the numerous friends who have so generously helped us in various ways in the preparation of the book: Dr. S. Brodetsky, Reader in Applied Mathematics, University of Leeds; Mr. A. Dakin, Headmaster of Stretford School, Manchester, and late Head of the Mathematical Department, Hymers College, Hull; Mr. R. C. Fawdry, Head of the Military and Engineering Side, Clifton College; Mr. C. W. Gilham, Assistant-Lecturer in Mathematics, University of Leeds; Dr. Charles M'Leod, Senior Mathematical Master, Aberdeen Grammar School; Mr. John Milne, Senior Mathematical Master, Mackie Academy, Stonehaven; Professor H. T. H. Piaggio, Department of Mathematics, University College, Nottingham; Mr. John W. Robertson, Head of the Mathematical Department, Central School, Aberdeen.

W. P. M.
G. J. B. W.

July 1920.

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HISTORICAL INTRODUCTION

For a long time a fierce and bitter controversy raged as to whether priority in the invention of the Calculus belonged to the Englishman Newton or to the German Leibnitz. There was practically universal agreement among mathematical historians that both these men reached the fundamental idea independently. Recent historical research, however, carried out by Mr. J. M. Child, gives the honour of first reaching the fundamental idea of the Calculus to Dr. Isaac Barrow, Professor of Mathematics at Cambridge, and Sir Isaac Newton's teacher. The following are Mr. Child's own words: "*Isaac Barrow was the first inventor of the Infinitesimal Calculus; Newton got the main idea of it from Barrow by personal communication; and Leibnitz also was in some measure indebted to Barrow's work, obtaining confirmation of his own original ideas and suggestions for their further development, from the copy of Barrow's book that he purchased in* 1673."

The general conclusion seems to be that Barrow reached his results in geometrical form, while Newton and Leibnitz developed Barrow's ideas analytically by means of algebra.

For students who are not concerned with the circumstantial and controversial minutiae of historical research, the broad facts of the invention of the Calculus stand out sharp and clear. We owe the invention of the Calculus in the first instance to the pioneering genius of the three men, Barrow, Newton and Leibnitz. We shall therefore give some account of their lives and careers.

Isaac Barrow was born in 1630, the son of a linen-draper in London. He died in 1677, and was buried in Westminster Abbey. He was first of all sent to the Charterhouse School, and there did not develop at all in the way that *ex cathedra* moralists prescribe for youth. He hated lessons, loved fighting, and altogether was such a worry to his staid old father that the latter was heard to pray that "if it pleased God to take one of his children, he could best spare Isaac." He was subsequently sent to Felstead School. Later, he turned over a new leaf, devoted himself enthusiastically to his studies, and was entered first at Peterhouse, and later on at Trinity College, Cambridge. Here he devoted himself to many subjects, literature, natural philosophy, anatomy, botany and chemistry. He then went on a tour through Europe—as was the custom of many of the best-educated and enterprising young men of his day—journeying through France and Italy to Asia Minor, returning home by Constantinople, Venice, Germany and Holland. It was at Constantinople that he first met with and studied the works of St. Chrysostom, which turned his thoughts to divinity. He returned to England, was ordained, and at the age of thirty became Professor of Greek at Cambridge. A few years later he was appointed Professor of Geometry at Gresham College, and in the following year was elected to the Lucasian Professorship of Mathematics at Cambridge, in which capacity he met and influenced Newton, whom he examined in Euclid for a mathematical scholarship at Trinity College. He ultimately resigned the Chair in favour of Newton, whose superior abilities he frankly recognised. He became Master of Trinity in 1672, and devoted the remainder of his life to the study of Divinity. Barrow, besides being a distinguished scholar, was noted for his courage and wit. In personal appearance he was small in size, lean and pale. He was a great favourite of Charles II., but was not a *persona grata* with the courtiers of the day, who were for the most part unable to form any distinct views on his mental abilities and achievements, but who could and did form very decided opinions on his slovenliness in dress, inveteracy as a smoker, and general untidiness.

The effect which his mathematical studies produced on his theological views is seen very clearly in the following prayer, which must rank as one of the most original and novel contributions to the theological literature of all time :

ISAAC BARROW'S PRAYER.

God always acts Geometrically.

How great a Geometrician art Thou, O Lord! For while this Science has no Bound: while there is for ever room for the Discovery of new Theorems, even by Human Faculties, Thou art acquainted with them all at one View, without any train of Consequences, without any wearisome Application of Demonstrations. In other Arts and Sciences our Understanding is able to do almost nothing; and, like the Imagination of Brutes, seems only to dream of some uncertain Propositions: Whence it is that in so many men are almost so many minds. But in these Geometrical Theorems all Men are agreed: In these the Human Faculties appear to have some real Abilities, and those Great, Wonderful and Amazing. For those Faculties which seem of almost no force in other Matters, in this Science appear to be Efficacious, Powerful and Successful, etc. Thee therefore do I take occasion to Love and Rejoice in, and Admire; and to pant after that Day, with the Earnest Breathings of my Soul, when Thou shalt be pleased, out of Thy Bounty, out of Thy Immense and Sacred Benignity, to grant me the favour to perceive, and that with a pure Mind and clear Vision, not only those Truths, but those also which are more numerous, and more important; and all this without that continual and painful Application of the Imagination which we discover these withal, etc.

Isaac Newton was born on Christmas Day, 1642, in the manor house of Woolsthorpe, six miles south of Grantham, and died in 1727. He is buried in Westminster Abbey. His entry into the world was not propitious. His mother was a widow, and he himself was a sickly child that seemed doomed to an early grave. So weak was he that two women who were sent to fetch physic for him from North Witham hardly expected to find him alive on their return. However, he grew to a fairly robust manhood, and seemed able to maintain the physical effort of his great mental activity without undue strain. As far as heredity and descent are concerned, young

Newton came of a most vigorous and wholesome stock. His father was a yeoman farmer, who owned the land he tilled, and whose forbears had done so for a century or more before him. His mother could be described in modern colloquial phrase as " not brilliant but thoroughly sound." She was a stout-hearted, industrious woman of the middle-class, who, after the death of her first husband, was recommended by a parishioner to the Rev. Barnabas Smith in search of a wife as " the widow Newton—an extraordinarily good woman." Isaac was sent first of all to a couple of village schools and thence for three years to the Grantham Grammar School. The instruction at school in those days consisted almost entirely of literary studies, and Newton did not take kindly to languages. He had little or no interest therefore in his school studies, and was not industrious. In fact, he tells us that he was the last boy in the lowest class but one. He was much more interested in manufacturing kites, windmills and water-wheels, at which he was an adept. He also tied paper lanterns to the tails of kites to make the country folk think in the darkness that a great comet had arrived which was like to swallow up the earth. His later career at school was, however, more propitious He managed to beat in a stand-up fight a boy bigger than himself, and this awoke in him the spirit of rivalry in other directions, so that ultimately he rose to be head of the school. On leaving school his mother, mindful of the agricultural stock from which he was sprung, desired to give him the requisite training as a practical farmer with a view to his efficiently cultivating his ancestral estates. Young Isaac could not be induced to take the slightest interest in turnips and cattle. He used to start from home as if to accompany an old family servant on marketing expeditions to Grantham, but he left his companion to act as plenipotentiary in transacting the business while he himself read books or thought out problems in an old attic or under a hedge by the roadside till the ambassador's return. He successfully manufactured a water-clock and a sun-dial, and otherwise made rapid though undisciplined advances in his intellectual development. His mother at length despaired of ever turning him into a farmer, and acting on the advice of her brother, the rector of Burton Coggles, she sent him to Cambridge, where he rapidly established himself as one of the very greatest, if not the greatest, scientific man that ever lived. Geometry, Astronomy, Gravitation, Optics, Pure Mathematics, all claimed his attention, and he made epoch-making discoveries in each and all. Dr. Isaac Barrow, at that time held the Lucasian Professorship of Mathematics at Cambridge, and Newton was in constant intercourse with him. The one mind re-acted consciously and unconsciously on the other, leading among other things to their epoch-making invention of the Calculus. Newton succeeded Barrow in this Chair, but later on the Government of the day made him Master of the Mint at a salary of £1,200 a year. He did the work with great conscientiousness and success. Sir Oliver Lodge's remarks on this episode in Newton's life are worth recording. " But what a pitiful business it all is ! Here is a man sent by Heaven to do certain things which no man else could do, and so long as he is comparatively unknown he does them ; but so soon as he is found out, he is clapped into a routine office with a big salary : and there is, comparatively speaking, an end of him. . . . It is for a great man's contemporaries to discover him, to make much of him, and to put him in surroundings where he may flourish luxuriantly in his own heaven-intended way." Of Newton's entire abstractedness and absorption in his thoughts while at Cambridge, numberless stories are told. He would get up in the morning, sit on the edge of the bed half-dressed and remain like that till dinner-time. One day an intimate friend called and found dinner on the table. Newton was long in coming, so the friend removed the cover, found a cooked chicken beneath, ate it, and replaced the cover with the bones underneath. At last Newton did appear, sat down to dinner, lifted the lid and casually remarked, " Dear me, I thought I had not had my dinner, but I see I have." On another occasion he got off his horse on a steep hill when riding home from Grantham. At the top he paused and turned round to re-mount his steed, when he found that he had only the bridle in his hand, the animal having slipped out its head and gone off on its own. When entertaining guests he often left the table to fetch a bottle of wine and forgot to return. He was slovenly in dress, languid in manner and so absent-minded as to be anything but a lively companion. He took no bodily exercise, had no amusements, and worked incessantly.

Newton's own idea of his life's labours may be quoted in his own words. "I know not what the world will think of my labours, but to myself it seems that I have been but as a child playing on the sea-shore; now finding some pebble rather more polished, and now some shell rather more agreeably variegated than another, while the immense ocean of truth extended itself unexplored before me."

GOTTFRIED WILHELM LEIBNITZ or LEIBNIZ was born in 1646 at Leipsig, where his father was professor of moral philosophy, and died at Hanover in 1716. His father died before he was six, and he therefore lost the advantage of a careful school education. His industry and natural ability, however, triumphed over all difficulties, and he soon read through all the German books at his command and learned Latin through the medium of the *Thesaurus Chronologicus* of Calvisius and an illustrated edition of Livy. At the age of fifteen, he entered the University of Leipsig as a law student. His first studies were devoted to law and philosophy, but later on he resolved to study mathematics. It was not till the summer of 1663 that he was able to study mathematics properly under a competent teacher, and he did so first of all in Jena under E. Weigel and later on at Paris, when he came under the direct influence of Huygens. In 1666 he was refused the degree of doctor of law at Leipsig, but it was promptly conferred on him the same year by the University of Altdorf in the free city of Nuremberg, where the brilliance of his dissertation led to the immediate offer of a professorial chair—an honour which he declined, as it did not fit in with his future schemes. Leibnitz now turned his attention actively to political life, but, throughout all his political career, he never abandoned his scientific and philosophical studies. So rapid and concentrated were his powers of mental application that he was able to maintain his mathematical trends of thought while occupied with political problems and diplomatic missions. He entered the service of the Elector of Mainz, and it fell to his lot to support the claims of the German candidate to the crown of Poland which had fallen vacant. A Polish prince was, however, elected to fill the vacant throne, despite the efforts of Leibnitz and his German colleagues. In 1672 Leibnitz went to Paris on the invitation of Louis XIV., who wished to discuss political questions with him. The French king, however, never granted Leibnitz the expected interview, but though politically the visit came to nothing, it was at Paris that he met Huygens, and their meetings started him on the study of geometry, which, according to his own account, opened for him a new world. In 1676, at the duke's request, he settled at Hanover, and spent the next forty years therein under the patronage of three successive princes, in charge of the ducal library. Later on in life began his connexion with Berlin, where he planned and founded the Akademie der Wissenschaften, with himself as its president for life. He was also held in high honour and esteem by Peter the Great of Russia, to whom he gave a plan for the establishment of an academy at St. Petersburg, a scheme which was carried out after the Czar's death. After many wanderings in Italy, Austria and Germany, Leibnitz returned to Hanover in 1714, but found that the elector had already gone to Great Britain as King George I. He died in 1716, his last years having been embittered by ill-health and neglect. Neither at Berlin, in the Academy which he had founded, nor in London, where his former elector had gone to rule, was any notice taken of his death, and he is described by the eye-witness, John Ker of Kersland, as having been buried at Hanover, with his secretary, Eckhart, as his only mourner, "more like a robber than what he really was, the ornament of his country." Only in the French Academy was his loss truly recognised and honoured. There is now a statue to him in Leipsig, his native town. The range of Leibnitz's mental powers was amazing. Mathematics, natural science, philosophy, theology, history, jurisprudence, politics, economics and philology, all received a share of his vigorous attention, and almost all of them were enriched by his original investigations. To the end of his life he preserved his unquenchable energy and powers of work in all the departments of human activity in which he interested himself. He had a marvellous power of carrying on his mind long and involved trains of scientific reasoning, whatever he happened to be doing, travelling, eating, etc. He was quick of temper, but charitable in his attitude towards those who differed from him in opinion. He was fond of money, keenly desirous of public honour, and felt bitterly the neglect in which his last years were spent. He stands out as one of the greatest scientific men of all time.

Those who would like further information on the development of the Calculus and its applications are strongly recommended to read Rouse Ball's *History of Mathematics at Cambridge.* There they will find amongst other things an account of the struggle which the Analytical Society carried on at Cambridge to overthrow in that University the notation $\dot{x}$ of Newton in favour of that of Leibnitz, namely, $\frac{dx}{dt}$, leading to Babbage's famous remark that the Analytical Society existed to advocate "the principles of pure **d**-ism as opposed to the **dot**-age of the university."

Brook Taylor (1685–1731), the son of John Taylor, of Bifrons House, Kent, and Olivia, daughter of Sir Nicholas Tempest, of Durham, was born at Edmonton in Middlesex. He studied mathematics at St. John's College, Cambridge, and took the degree of LL.D. in 1714. Taylor published in London, 1715, his *Methodus Incrementorum Directa et Inversa,* which contained, amongst other ingenious things, his famous result known as "Taylor's Theorem." The significance of this result was not realised till 1772, when Lagrange recognised its great importance and styled it "le principal fondement du calcul différentiel." Taylor was elected a Fellow of the Royal Society in 1712, and in the same year sat on the committee appointed to decide upon the rival claims of Newton and Leibnitz as to which had the priority of claim to be considered the inventor of the Calculus. From 1715 he became interested in religious and philosophical questions. He left two unfinished treatises, *On the Jewish Sacrifices* and *On the Lawfulness of eating Blood.* In 1721 he married a Miss Brydges, of Wallington, Surrey, and this led to an estrangement with his father, a man of morose temper. The lady died, however, in 1723, in giving birth to a son, who also died, as a result of which Brook Taylor returned to his ancestral home at Bifrons and spent the next two years with his own people. In 1725 he married again—this time with his father's approval—Sabetta Sawbridge, of Olantigh, Kent. She also died, in 1730, in giving birth to a daughter, who, however, survived. As a result of these repeated calamities, Taylor's none too robust health gave way, and he died at Somerset House in 1731, and was buried at St Ann's, Soho. As a mathematician he ranks very high, and was said in his own day to be the only English mathematician after Sir Isaac Newton and Roger Cotes capable of standing up to the Bernoullis. A great deal of the mathematical influence he might have had in his lifetime was lost, however, owing to the obscurity with which he wrote out his papers and stated his discoveries.

Colin Maclaurin (1698–1746) stands out as one of the greatest of the Scottish mathematicians. He was born at Kilmodan, in Argyllshire, where his father was parish minister. In 1709, at the early age of eleven years, he entered Glasgow University and came under the influence of Professor Robert Simson, the celebrated geometer and editor of Euclid. In 1717 the Chair of Mathematics at the Marischal College of Aberdeen—the College of Dugald Dalgetty in Sir Walter Scott's *Legend of Montrose*—fell vacant, and Maclaurin was the successful candidate as the result of a competitive examination. In 1719 he was elected a Fellow of the Royal Society and made the acquaintance of Newton. In 1722 and later Maclaurin travelled on the Continent as tutor and travelling companion to the eldest son of Lord Polwarth, during which sojourn he not only saw new scenes and cities but carried out a large amount of mathematical work. A thesis on the percussion of bodies obtained the prize of the French Academy of Sciences for the year 1724, and in 1725 he was elected Professor of Mathematics at the University of Edinburgh, his candidature having received the strongest support of Newton. In 1745 Bonnie Prince Charlie had landed on the west coast of Scotland and was in full march with his Highland adherents against the city of Edinburgh. The citizens of the Scottish capital were in terror at the approach of the Highlanders, whose reputation for courage and ferocity had been traditional in Scotland from time immemorial. They hastily set themselves to defend the city, and improvised trenches and barricades to stay the Highland onrush. In his *Tales of a Grandfather* Sir Walter Scott says: "In the meantime, fortifications were added to the walls, under the scientific direction of the celebrated M'Laurin, Professor of Mathematics in the University of Edinburgh." The result of Maclaurin's exertions in preparing trenches and obstacles against the approach of the Highland host, and of the cold and privation which he endured, was the sowing of the seeds of dropsy, to which disease he afterwards succumbed. As soon as the Prince got possession of Edinburgh, Maclaurin fled to England and remained with Archbishop Herring of York till it was safe to return to Edinburgh. He died in the Scottish capital in 1746, the year of the battle of Culloden and of the crushing of the second Jacobite Rebellion. Maclaurin's researches covered many branches of mathematics—Analysis, Geometry, and Applied Mathematics.

CHAPTER I.

USES OF SQUARED PAPER: RATES, VELOCITIES, ACCELERATIONS.

1. Relative Rates and Gradients.

The fundamental problem of the Calculus is to find the rate at which one quantity varies with respect to another. We shall therefore discuss first of all the graphical representation of such rates.

CASE I. LINEAR GRAPHS.

The simplest case is where the graph connecting one quantity with another is a straight line. Consider the two points P and Q on the straight line UV in the above diagram. If the coordinates of any point, say P, are x and y,

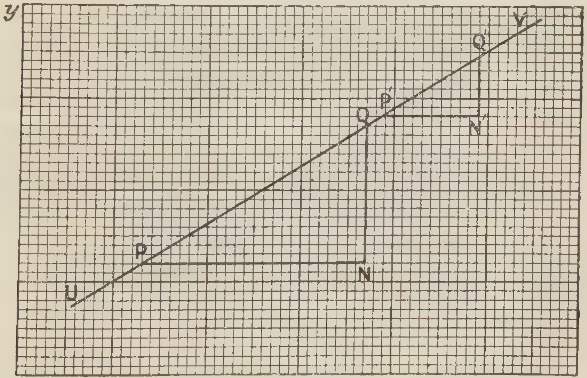


FIG. 1.

we shall write shortly $P \equiv (x, y)$. It will be remembered that the x-coordinate is called the "abscissa" of P and the y-coordinate the "ordinate" of P.

We may regard the above graph as representing to the eye the variations of y corresponding to those of x. Also, if we proceed along the graph from P to Q, the increase of the "ordinate" of Q over that of P is NQ; while the increase of the "abscissa" of Q over that of P is PN.

Hence the ratio of the increase of the ordinate of Q over that of P to the increase of the abscissa of Q over that of P is $\frac{NQ}{PN}$.

If we take any other two points P′, Q′ on the straight line UV, we see that the ratio of the difference of the ordinates of P′ and Q′ to the difference of their abscissae is $\frac{N'Q'}{P'N'}$.

Now by similar triangles $\frac{NQ}{PN} = \frac{N'Q'}{P'N'}$. Hence the ratio $\frac{NQ}{PN}$ is constant in value for all positions of P and Q on the line UV.

Definition. *The ratio of* $\frac{NQ}{PN}$, *which is constant for all positions of* P *and* Q *on the given linear graph, is called the "gradient" of the given linear graph.*

We have therefore the following proposition: *If the graph connecting two variable quantities x and y is a straight line, the gradient of this line measures the rate at which y increases with respect to x.*

The following are examples of the practical use of this principle in plotting one quantity against another on squared paper.

Example 1. *A train is going at the rate of* 40 *miles an hour. Represent this graphically.*

The following chart is easily calculated:

Time in hours - -	0	1	2	3	4	5	6	7	8
Distance in miles traversed	0	40	80	120	160	200	240	280	320

Let us take any two convenient points P and Q on the subjoined graph. Then it is plain that, as the train traverses the distance between the two positions represented on the graph by P and Q,

$$\frac{\text{Distance traversed}}{\text{Time taken}}=\frac{NQ}{PN}\text{ miles per hour}=\frac{120}{3}\text{ miles per hour}$$

$$=40\text{ miles per hour}=\text{the speed of the train.}$$

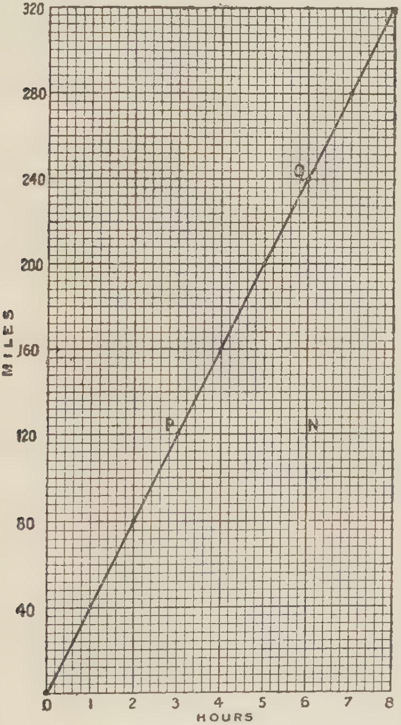


FIG. 2.

Thus we have verified in a particular case that the gradient represents graphically the rate at which the distance traversed by the train varies with respect to the time required, viz. 40

miles per hour, in passing between any two points of the train's route.

Example 2. *Two numbers x and y are so related as to have the following sets of values :*

	(1)	(2)	(3)	(4)
x	0·113	0·196	0·253	0·360
y	0·376	0·417	0·445	0·499

Show that x increases twice as fast as y.

With an obvious notation we have

$$x_1 = 0{\cdot}113, \qquad x_2 = 0{\cdot}196,$$
$$y_1 = 0{\cdot}376, \qquad y_2 = 0{\cdot}417\,;$$
$$\therefore\ x_2 - x_1 = 0{\cdot}083 \text{ and } y_2 - y_1 = 0{\cdot}041.$$

Hence $$\frac{x_2 - x_1}{y_2 - y_1} = 2 \text{ (very approx.).}$$

Similarly $$\frac{x_4 - x_3}{y_4 - y_3} = \frac{0{\cdot}107}{0{\cdot}054} = 2 \text{ (very approx.).}$$

We can proceed thus for every increment in the corresponding values of x and y, and so verify the given statement.

Example 3. *The subjoined chart from Whitaker's Almanack gives the times of sunrise on the respective dates during April,* 1913. *Plot these on squared paper, and hence "read off" the average rate at which the time of sunrise is becoming earlier.*

APRIL, 1913.

Date - -	1	2	3	4	5	6	7	8	9	10
Time of Sunrise	5.37	5.35	5.33	5.31	5.29	5.27	5.24	5.22	5.20	5.18
Date - -	11	12	13	14	15	16	17	18	19	20
Time of Sunrise	5.15	5.13	5.11	5.8	5.6	5.4	5.2	5.0	4.58	4.56
Date - -	21	22	23	24	25	26	27	28	29	30
Time of Sunrise	4.54	4.52	4.50	4.48	4.46	4.44	4.42	4.39	4.37	4.35

We note that the points lie approximately on a straight line. We therefore draw that straight line which appears to the eye to lie most evenly amongst the plotted points. We then take two convenient points P and Q on the straight line thus obtained.

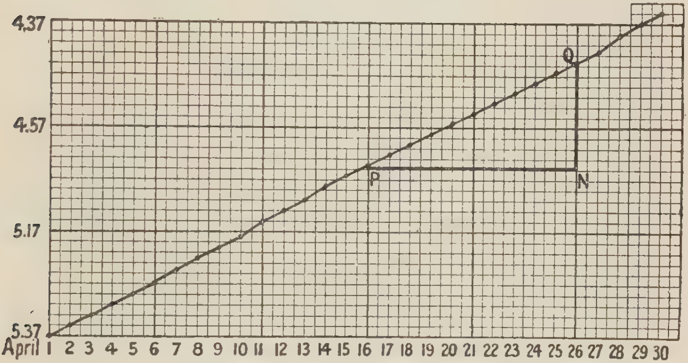


FIG. 3.

Then in PN days, the time of sunrise has become NQ minutes earlier.

∴ average rate at which the time of sunrise is becoming earlier

$$= \frac{NQ}{PN} \text{ min. per day} = \tfrac{20}{10} \text{ min. per day} = 2 \text{ minutes per day.}$$

CASE II. CURVED GRAPHS.

The next case deals with the method of measuring graphically the rate at which one quantity varies with respect to another when the plotted points lie on a curved graph.

Consider the following example :

The following gives the population of England and Wales at each Decennial Census from 1801 *to* 1911. *Plot as accurately as you can on squared paper, and hence "read off" the approximate average rate at which the population was increasing during the intervals* 1841-1861 *and* 1801-1811.

Year.	Population.	Year.	Population.
1801	8,892,536	1861	20,066,224
1811	10,164,256	1871	22,712,266
1821	12,000,236	1881	25,974,439
1831	13,896,797	1891	29,002,525
1841	15,914,148	1901	32,527,843
1851	17,927,609	1911	36,075,269

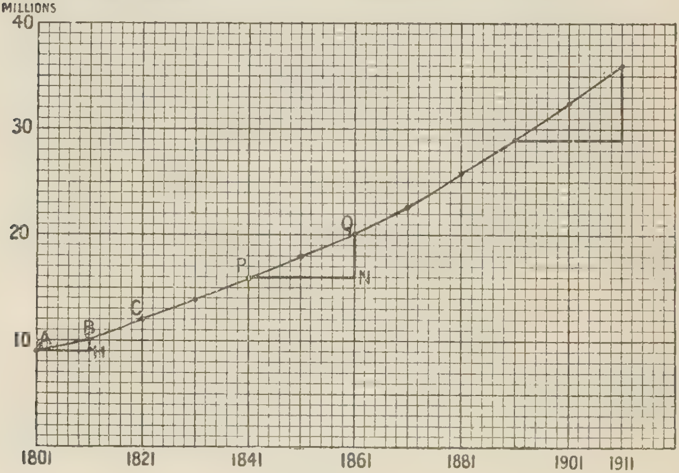


FIG. 4.

During the interval 1841-1861 we may treat the arc PQ as very approximately a straight line. Hence the average rate of increase of the population during the interval 1841-1861 is measured by $\frac{\text{NQ (people)}}{\text{PN (years)}}$, *i.e.* by 4 millions per 20 years, *i.e.* by 200,000 persons per annum.

Again, if we take the initial interval where the graph is curving more rapidly, we take a shorter portion AB of the graph so as to get the arc more nearly a straight line. The average rate of increase of the population during the interval 1801-1811 is then measured by $\frac{\text{HB (people)}}{\text{AH (years)}}$, *i.e.* by 1 000,000 per 10 years, *i.e.* by 100,000 persons per annum.

We thus see that where the graph curves more and more rapidly, we take a shorter and shorter portion of the arc so as to get as nearly as possible a straight line and measure its gradient.

But if we join the two points A, B by a straight line and produce it both ways, we see that the straight line thus obtained is very approximately a tangent to the curve, and its gradient is the same as the gradient of the arc AB treated as a straight line. Hence we obtain the following result, which is of the utmost importance:

In the limit, the gradient of the tangent at a point on the graph represents the rate at which the one quantity is varying with respect to the other at that particular point on the graph, due regard being had to the units employed.

A few examples of the practical use of this method will now be given. We draw the tangent at the particular point of the graph under consideration and "read off" the gradient of the tangent thereat:

Example 1. *The following table gives the times of sunrise and sunset of the respective dates at intervals of* 28 *days throughout the year* 1913. *Plot on squared paper.*

At what rate is the length of the daylight increasing or decreasing on Dec. 31*st, March* 26*th, June* 7*th, Nov.* 16*th, Dec.* 31*st*?

Date - -	Jan. 1	Jan. 29	Feb. 26	Mar. 26	Apr. 23	May 21	June 18
Time of Sunrise	8.8	7.46	6.56	5.51	4.50	4.2	3.44
Time of Sunset	3.58	4.40	5.30	6.21	7.7	7.50	8.18

Date - -	July 16	Aug. 13	Sept. 10	Oct. 8	Nov. 5	Dec. 3	Dec. 31
Time of Sunrise	4.3	4.44	5.28	6.14	7.2	7.48	8.8
Time of Sunset	8.9	7.26	6.26	5.22	4.26	3.52	3.58

Draw the tangents at the points on the graph corresponding to the given dates. (See Fig. 5.)

Increase on March 26th is $\frac{N_1Q_1 \text{ (min.)}}{P_1N_1 \text{ (days)}}$, *i.e.* 264 min. per 67 days, or about 4 min. per day.

Increase on June 7th is $\frac{N_2Q_2 \text{ (min.)}}{P_2N_2 \text{ (days)}}$, *i.e.* 132 min. per 73 days, or about 2 min. per day.

Decrease on Nov. 16th is $\frac{N_3Q_3 \text{ (min.)}}{P_3N_3 \text{ (days)}}$, *i.e.* 204 min. per 78 days, or about $2\frac{1}{2}$ min. per day.

On Dec. 31st there is zero rate of increase or decrease, because the tangent of the graph is parallel to the date axis, and consequently in this case the "NQ" of the "PNQ" gradient triangle vanishes.

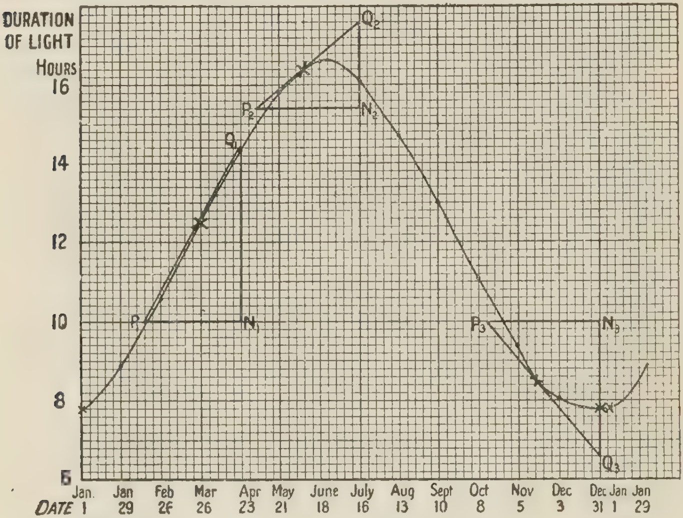


FIG. 5.

Example 2. *A ball rolls down an inclined plane and the following observations are made, t being the time in seconds since the beginning of motion and s being the distance in feet rolled down. Find the rate at which the ball is rolling (a) after* 1 *second, (b) after* 2 *seconds, (c) after* $3\frac{1}{2}$ *seconds, (d) after* 3·8 *seconds.*

t	0·0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0
s	0·0	0·2	0·8	1·9	3·3	5·2	7·5	10·3	13·4

Draw the tangents at the points on the graph corresponding to the given instants of time. (See Fig. 6.)

After 1 sec. the rate of rolling is Q_1M_1 ft. in R_1M_1 sec. $= \frac{8}{3}$ ft. per $\frac{3}{2}$ sec. $= \frac{16}{9}$ ft. per sec. $= 1{\cdot}8$ ft. per sec.

After 2 sec. the rate of rolling is Q_2M_2 ft. in R_2M_2 sec. $=\frac{25}{3}$ ft. per $\frac{13}{5}$ sec. $=\frac{125}{39}$ ft. per sec. $=3{\cdot}2$ ft. per sec.

After $3\frac{1}{2}$ sec. the rate of rolling is Q_3M_3 ft. in R_3M_3 sec. $=\frac{31}{3}$ ft. per $\frac{17}{10}$ sec. $=\frac{310}{51}$ ft. per sec. $=6{\cdot}1$ ft. per sec.

After $3{\cdot}8$ sec. the rate of rolling is Q_4M_4 ft. in R_4M_4 sec. $=\frac{17}{3}$ ft. per $\frac{9}{10}$ sec. $=\frac{170}{27}$ ft. per sec. $=6{\cdot}3$ ft. per sec.

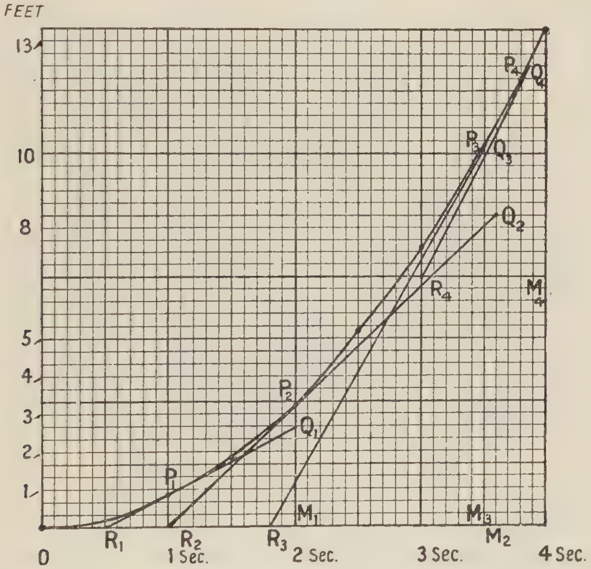


FIG. 6.

Example 3. *x and t are the distance in miles and the time in hours of a train from a railway station. Plot on squared paper. State how the curve shows where the speed is greatest and where it is least.*

t	0	0·05	0·10	0·15	0·20	0·25	0·30	0·35	0·40	0·45	0·50
x	0	0·25	1·00	3·05	5·00	5·85	6·10	6·10	6·35	7·00	7·65

(Bd. of Edu.)

The rate of displacement or speed is proportional to tan ψ, where ψ is the inclination of the tangent to the axis of x.

Hence the speed is small where ψ is small and great where ψ is great.

Thus the speed at O is small and gradually increases to between A and B. Then it decreases and becomes very small at D, remaining so until it approaches E, when it again increases.

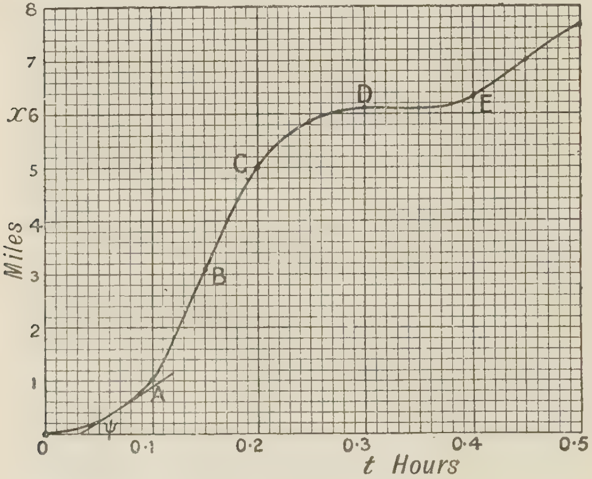


FIG. 7.

2. Acceleration.

In nature, bodies rarely move with the same velocity throughout their motion. If the velocity be not uniform, it must be either increasing or decreasing. If the velocity of a body be changing, the body is said to possess "acceleration," or its motion is said to be "accelerated." *The rate of change of the velocity of a body is called its "acceleration."*

If the velocity of a body is increasing uniformly at the rate of 3 ft. per sec., every second, the body is said to possess an acceleration of three feet per second per second, *i.e.* 3 ft. per sec. per sec. or 3 ft. $(\text{per sec.})^2$.

Since the acceleration is the rate of change of the velocity, the gradient of the tangent to the velocity-time graph will give the acceleration at any instant.

Example. *The following table gives the distances moved down a hill by a rolling ball in the corresponding times from the start of the motion. Find the velocity and acceleration after 3 seconds.*

t sec.	0	1	2	3	4	5	6
s feet	0	13	37·4	73·2	120·4	179	249

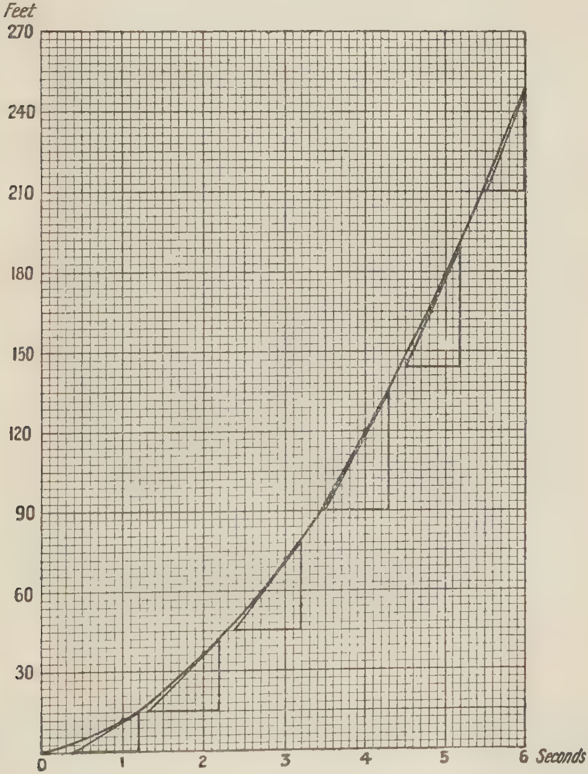


FIG. 8.

Plotting s against t as in the above diagram and drawing tangents to the space-time graph, we "read off" the following table of corresponding velocities and times (the gradients of the various tangents giving the required velocities, *i.e.* the values of v):

t seconds	1	2	3	4	5	6
v ft. per sec.	19	30	41·4	52·5	64	76

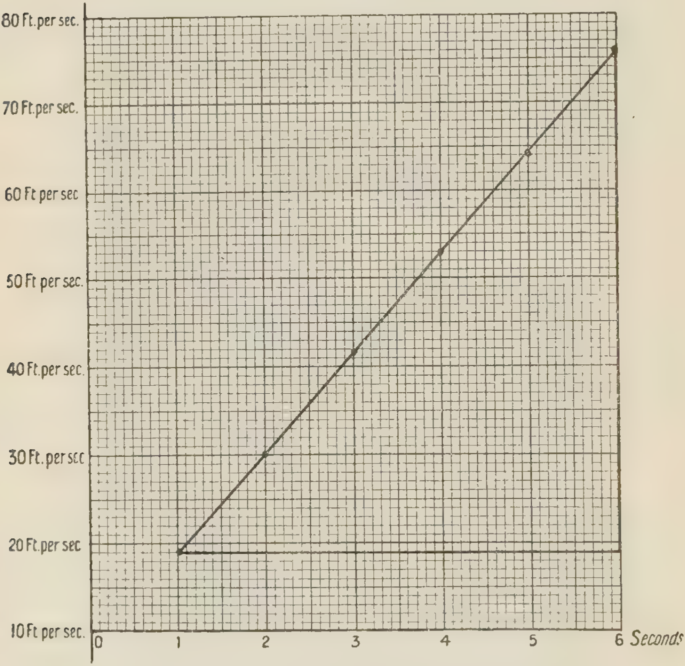


FIG. 9.

Again, plotting v against t in the above diagram from the values in the foregoing table, we see that the velocity-time graph is a straight line showing that the acceleration is constant and equal to 57 ft. per sec. in 5 sec. or 11·4 ft. per sec. per sec. Moreover the velocity after 3 seconds is 41·4 ft. per sec.

EXAMPLES I.

1. A particle is projected up a smooth inclined plane which is held at such an angle that the distance traversed from the lower edge of the plane is given by $s = 10t - t^2$, where s is measured in feet and t in seconds. Plot s against t, and hence obtain the time when the particle ceases to ascend and begins to descend.

2. In Qu. 1, find the velocity after 2 seconds, by drawing the tangent to the graph.

3. In Qu. 1, find the velocity after 8 seconds.

4. Show from the graph in Qu. 1 that the velocity with which the particle passes any point in the ascent is the same as that with which it passes the same point in the descent.

5. Plot the following values for s and t:

t sec.	0	1	2	3	4	5	6	7	8	9	10
s ft.	3	4·5	6	7·5	9	10·5	12	13·5	15	16·5	18

(1) What does the graph tell you about the motion ?
(2) Read off the velocity after 3, 5, 8 seconds.

6. Plot the following values for s and t:

t sec.	0	1	2	3	4	5	6	7	8	9	10
s ft.	−4	−2·5	−1	0·5	2	3·5	5	6·5	8	9·5	11

(1) What does the graph tell you about the motion ?
(2) "Read off" the velocity.

7. Plot the following values for s and t:

t	0	1	2	3	4	5	6	7	8	9	10
s	−1	−2·5	−4	−5·5	−7	−8·5	−10	−11·5	−13	−14·5	−16

(1) Show from your graph that the velocity is uniform.
(2) "Read off" the velocity.

8. A fast train from Bristol to London passes the stations at the following times :

	Times.	Distance in miles.
Bristol, -	- 2.11	0
Bath, -	- 2.35	$11\frac{1}{2}$ miles.
Reading, -	- 3.52	$81\frac{1}{2}$ miles.
London, -	- 4.30	$118\frac{1}{4}$ miles.

Assuming that the train stops for a very short time at the stations, draw the graph and *thence* deduce the "average speed" with which the train performs this journey.

9 As in Qu. 8, find the "average velocity" (by means of squared paper) with which the train performs the following journey :

	Times.	Distance in miles.
Taunton, -	- 6.40	$44\frac{3}{4}$
Durston, -	- 6.52	39
Bridgwater,	- 7.5	$33\frac{1}{4}$
Dunball, -	- 7.12	$30\frac{3}{4}$
Highbridge,	- 7.26	27
Bristol, -	- 8.50	0

10. A point has the following velocities in feet per second at the following times :

Velocity	100	103·3	105·7	107·2	107·8
Time -	10	10·01	10·02	10·03	10·04

Find the approximate acceleration when t is 10·02.
(A graphical method is best.) (Bd. of Educ.

11. A piece of mechanism moves so that the distance x mm. of a certain point in it from a fixed point after t seconds is given by the following table :

t sec.	0	1	2	3	4	5
x mm.	0	25	82	177	316	505

Find the velocity of the moving point after 1, 2, 3, 4, 5 seconds, and thence the acceleration of the point after 3 seconds.

12. x and t are the distance in miles and the time in hours of a train from a railway station. Plot on squared paper. Describe clearly why it is that the *slope* of the curve shows the speed. Where is the speed greatest, and where is it least ?

x	0	0·12	0·50	1·52	2·50	2·92	3·05	3·05	3·17	3·50	3·82
t	0	0·05	0·10	0·15	0·20	0·25	0·30	0·35	0·40	0·45	0·50

(Bd. of Educ.)

13. The following numbers give x feet the distance of a sliding piece measured along its path from a certain point to the place where it is at the time t seconds ; what (approximately) is its acceleration after 0·5 second ? Show by a curve how the acceleration depends upon t.

x	1·0	2·7	4·4	6·0	7·4	8·7	9·7	10·4	10·8	11·0
t	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9

(Adapted from Bd. of Educ.)

14. Plot the curve $y=3x^2-x^3$ as accurately as you can. Hence, by drawing the tangents, obtain the gradients at the points given by $x=0$, $x=1$, $x=2$. (Clifton.)

15. A body moving in a straight line describes the following distances in the given times :

Time in seconds	0	5	10	15	20	25	30
Distance in feet	0	13·75	60	176·25	400	768·75	1320

Draw approximately the space-time and the velocity-time curves for the same period, and find the acceleration after 20 seconds.
(Adapted from Cambridge.)

16. A cart going at the rate of 5 miles per hour passes a milestone, and 14 minutes afterwards a bicycle, going in the same direction at 12 miles per hour, passes the same milestone. Find when and where the bicycle will overtake the cart. (Clifton.)

17. Two men set out at the same instant, one to walk from Bath to Bristol, a distance of 10 miles, at 4 miles per hour, the other to walk from Bristol to Bath at $3\frac{1}{2}$ miles per hour. For how long do they walk before they meet ? (Solve graphically.) (Clifton.)

18. Find the accelerations by means of squared paper (*a*) after 2 seconds, (*b*) after 3·5 seconds, (*c*) after 4·7 seconds, from the following data :

t sec.	0·0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0	4·5	5·0
v ft. per sec.	0	16	32	48	64	80	96	112	128	144	160

19. The following table gives the time of sunset in the months of November and December 1913. Plot on squared paper, and hence find the rate at which the sun is setting earlier or later on the following dates : (*a*) Nov. 6, (*b*) Nov. 13, (*c*) Nov. 23, (*d*) Dec. 1, (*e*) Dec. 26 :

Nov. 1	Nov. 6	Nov. 11	Nov. 16	Nov. 21	Nov. 26	Dec. 1
4.33	4.24	4.16	4.9	4.3	3.57	3.53
Dec. 6	Dec. 11	Dec. 16	Dec. 21	Dec. 26	Dec. 31	
3.50	3.49	3.50	3.51	3.54	3.58	

20. An electric train is running at a uniform speed, and the brakes are applied when it is at A, one mile from a station B. The distance, s feet, of the train from A, at the end of t seconds, is given by the equation $s = 46t - \frac{t^2}{10}$. What is the velocity of the train 30 seconds after passing A ? Will it reach the station before it stops ?

CHAPTER II.

DIFFERENTIATION OF POSITIVE INTEGRAL POWERS OF x. MAXIMA AND MINIMA.

1. To find the gradient of the line joining the points $\mathbf{P} \equiv (\mathbf{x}, \mathbf{y})$ and $\mathbf{Q} \equiv (\mathbf{h}, \mathbf{k})$.

Draw PM and QN perpendicular to $x'x$.

Draw PK parallel to $x'x$.

Then plainly

$$\text{OM} = x \quad \text{and} \quad \text{MP} = y.$$
$$\text{ON} = h \quad \text{and} \quad \text{NQ} = k.$$

Now the gradient of the line is $\dfrac{\text{KQ}}{\text{PK}} = \dfrac{k-y}{h-x}$.

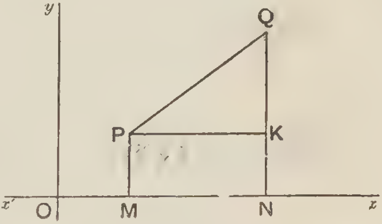


Fig. 10.

Example 1. *Find the gradient of the line joining the points* (3, 5) *and* (11, 19).

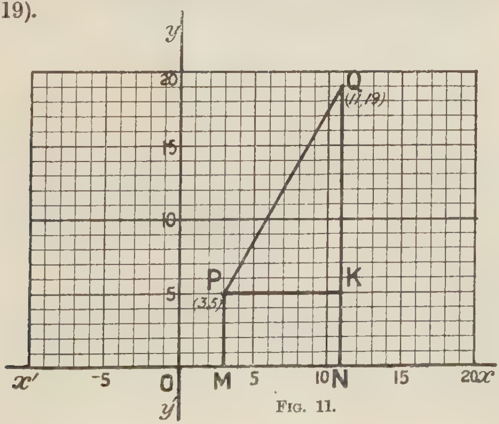


Fig. 11.

Making the construction as above, we have:

$$\text{Gradient of the line PQ is } \frac{KQ}{PK}=\frac{19-5}{11-3}=1{\cdot}75.$$

Example 2. *Find the gradient of the line joining the points* (−4, 9) *and* (6, −10).

In this case we have that the gradient of the line PQ is

$$\frac{KQ}{PK}=\frac{9-(-10)}{-4-(6)}=-\frac{19}{10}.$$

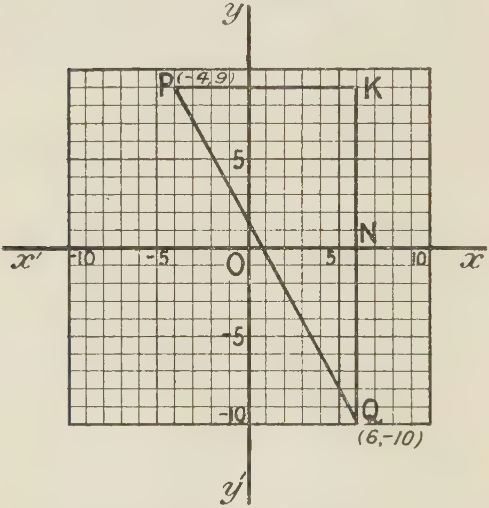


FIG. 12.

A little consideration will show us that when the line slopes from right top to left bottom the gradient is positive, and when the line slopes from left top to right bottom the gradient is negative.

2. Notation of δx and δy.

Often we find it convenient to use the word "increment" or "extra bit," in practical life. For example, the inhabitants of a certain town wished to erect a statue to Sir Walter Scott. They gathered £1000, but this was not enough, so they required an "increment" of £200.

Again, a certain family found that five pounds of butter per week were not enough, so they ordered in "a pound and a half extra."

In mathematics δx is used to denote an extra bit added on to x, δ being the Greek character for the first letter of the word "difference." (The English letter **d** is required later on to express a different shade of meaning.)

In the above cases,

$$\delta(\text{£}1000) = \text{£}200.$$

$$\delta(5 \text{ lb. of butter}) = 1\tfrac{1}{2} \text{ lb. of butter.}$$

Again, suppose that a man's watch is twenty minutes fast. Then, whatever time his watch indicates, he has to subtract 20 min. to get the correct time. Hence in this case,

$$\delta(\text{time}) = -(20 \text{ min.}).$$

Consider once more the subjoined diagram :

Let $x'x$ be a straight line and let O be a marked point thereon. Let P be situated at a distance x cms. to the right of O. Then

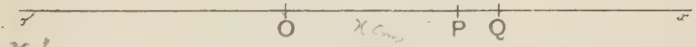


FIG. 13.

if Q lie some distance to the right of P, we should say (according to the above notation) that $PQ = (\delta x)$ cm.

$$\therefore \; OQ = (x + \delta x) \text{ cm.}$$

The meaning and uses of δx will now be plain.

Example 1. *The sides of a rectangular piece of metal plate are 7 cm. and 5 cm. The plate is heated so that the sides increase by* 0·2 *per cent. of their length. If* A *denote the area of the plate at first, find* δA *when the plate is heated.*

The sides at first are 7 cm. and 5 cm. ....(1)

The sides when heated are (7 + 0·2 per cent. of 7) cm. and (5 + 0·2 per cent. of 5) cm.,

i.e. 7·014 cm. and 5·010 cm. ....(2)

Hence, by (1) and (2),

$$\delta A = (7{\cdot}014 \times 5{\cdot}010 - 7 \times 5) \text{ sq. cm.}$$
$$= 0{\cdot}14 \text{ sq. cm. (correct to two places).}$$

Example 2. *If* $y=x^2$, *find the increase in* y *when* x *is increased to* $x+\delta x$.

Plainly
$$\delta y=(x+\delta x)^2-x^2$$
$$=2x\,.\,\delta x+(\delta x)^2$$

3. To find the gradient of the line joining the points (x, y) and (x + δx, y + δy).

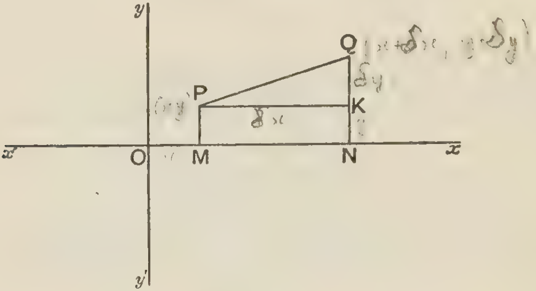


FIG. 14.

Let the points be $\quad P\equiv(x,\,y),$
$$Q\equiv(x+\delta x,\,y+\delta y).$$

Draw PM and QN each perpendicular to Ox.

Draw PK parallel to Ox.

Then $\quad PK=MN=\delta x$

and $\quad KQ=\delta y.$

Now there is a rise of distance KQ in the direction parallel to Oy corresponding to a distance PK described in the direction parallel to Ox.

Hence there is a rise of δy corresponding to δx,

i.e. the gradient is $\dfrac{\delta y}{\delta x}$.

Example. *Find the gradient of the chord joining the points whose x coordinates are 2 and 5 on the curve $y=x^2$.*

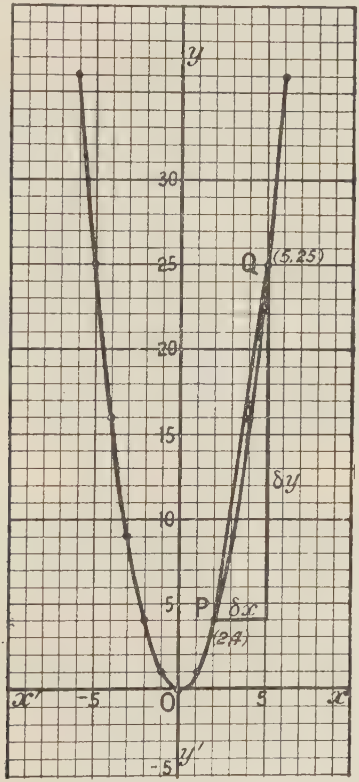


FIG. 15

When $x=2$, $y=4$;

when $x=5$, $y=25$.

$$\therefore\ \delta x=3 \quad \text{and} \quad \delta y=21.$$

$$\therefore\ \text{gradient}=\tfrac{21}{3}=7.$$

4. The tangent to a curve.

For the purposes of this discussion it is convenient to regard "the tangent at a point" to a given curve as defined in the following way:

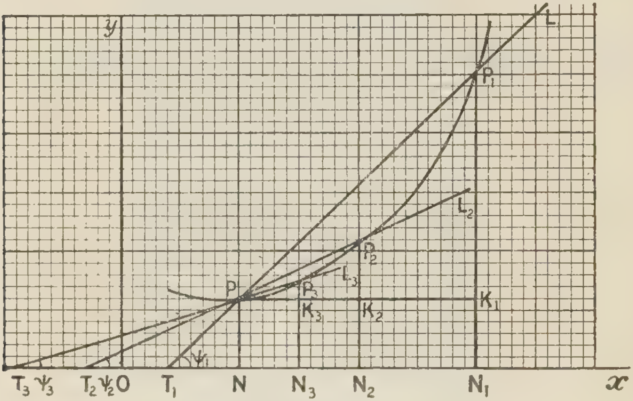


FIG. 16.

To obtain the tangent at the point P on the curve, take a point P_1 on the curve conveniently near to P. Complete the figure in the usual way, as shown in the diagram.

Next bisect the distance NN_1 by N_2.

Let P_2 be the point on the curve corresponding to N_2.

Complete the chord $T_2PP_2L_2$.

It is plain from the figure that $T_2PP_2L_2$ will be a nearer approximation to the tangent at P than the chord $T_1PP_1L_1$.

Next let N_3 be the mid-point of NN_2 and so on. This process is called the process of "*repeated bisection.*"

If we proceed in this way, it is plain that by continual bisection of the intervals NN_1, NN_2, NN_3, etc., we shall obtain chords which *in general* do not reach the position of the tangent at P but become as close approximations to it as we please, if we continue this process of "*repeated bisection*" sufficiently long.

We note that there are an indefinite number of other methods besides the method of "*repeated bisection*" by which we can make N_1, N_2, N_3, ..., etc., approach infinitely close to N, in position.

It is impossible for us within the limits of this book to investigate more closely the algebraic conception of the tangent to a curve. Let it suffice to say that by calculating the gradient of the chord PP_1, we are able from the algebraic expression thus found to *see* intuitively what the gradient will be when we are dealing with what is actually the tangent at P and not a mere approximation to it.

A few examples will make this plain.

Example. *To find the gradient of the tangent at any point on the curve* $y=x^2$.

Let x be the x-coordinate of the point P on the curve under investigation.

Let $x+\delta x$ be the x-coordinate of a point Q on the curve conveniently near to P.

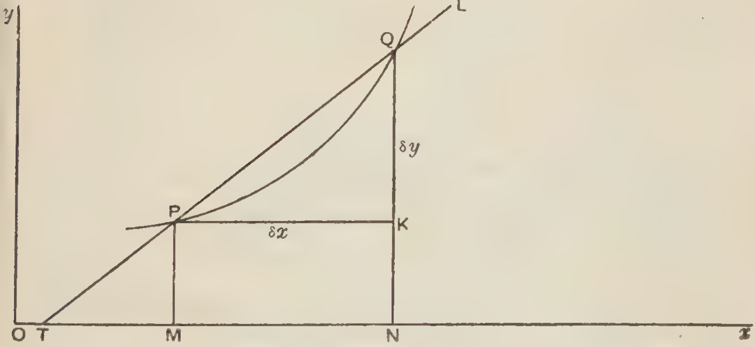


FIG. 17.

Draw the above figure as usual.

Then $$y=x^2$$

and $$y+\delta y=(x+\delta x)^2.$$

$$\therefore\ \delta y=(x+\delta x)^2-x^2$$
$$=2x\,.\,\delta x+(\delta x)^2.$$
$$\therefore\ \frac{\delta y}{\delta x}=2x+\delta x.$$

Now as N approaches M (by repeated bisection or otherwise), it is plain that the right-hand side approaches in value to $2x$, since δx becomes smaller and smaller as N approaches M.

Hence the gradient of the tangent at P is $2x$.

Similarly the student can easily show that if $y=x^3$,

$$\frac{\delta y}{\delta x}=3x^2+3x\,\delta x+(\delta x)^2,$$

and if $y=x^4$,
$$\frac{\delta y}{\delta x}=4x^3+6x^2\,.\,\delta x+4x(\delta x)^2+(\delta x)^3,$$

and hence that the corresponding gradients to the tangents will be $3x^2$ and $4x^3$ respectively.

5. Generalization of the results of Art. 4.

By direct calculation the student can obtain the following table:

Curve.	Gradient.	Curve.	Gradient.
$y=x$	1, *i.e.* x^0	$y=x^5$	$5x^4$
$y=x^2$	$2x$	$y=x^6$	$6x^5$
$y=x^3$	$3x^2$	$y=x^7$	$7x^6$
$y=x^4$	$4x^3$	etc.	etc.

The law of formation of the gradients of the successive curves is then plain.

In the case of the curve $y=x^n$, where n is a positive integer, the gradient of the tangent is nx^{n-1}.

6. Formal proof that the gradient of the tangent to the curve $y=x^n$ is nx^{n-1}, where n is a positive integer.

[We shall require the well-known algebraic theorem :

$$a^n-b^n\equiv(a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+ab^{n-2}+b^{n-1}).]$$

Definition. We shall use $\frac{\delta y}{\delta x}$ to denote the gradient of the chord joining the two points (x, y) and $(x+\delta x, y+\delta y)$ on a curve, and $\frac{dy}{dx}$ to denote the gradient of the tangent at the point (x, y) on the curve.

(i) *To find in terms of x and x_1 the gradient of the chord joining the points* $P \equiv (x, y)$ *and* $P_1 \equiv (x_1, y_1)$ *on the curve* $y = x^n$, *where* n *is a positive integer.*

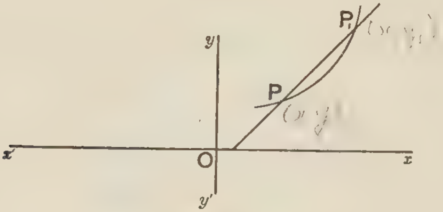


FIG. 18.

Plainly $\qquad y = x^n \quad$ and $\quad y_1 = x_1^n.$

$$\therefore\ y_1 - y = x_1^n - x^n$$

$$\equiv (x_1 - x)(x_1^{n-1} + x_1^{n-2}x + x_1^{n-3}x^2 + \dots + x^{n-1}).$$

$$\therefore\ \frac{\delta y}{\delta x} \equiv \frac{y_1 - y}{x_1 - x} = x_1^{n-1} + x_1^{n-2}x + x_1^{n-3}x^2 + \dots + x^{n-1},$$

with the usual notation.

(ii) *To deduce the gradient of the tangent at the point* P.

We allow the point P_1 to move up indefinitely close to P by the method of "*repeated bisection*" (or otherwise) as explained in Art. 4, that is, we put $x_1 = x$ in the formula of (i) just obtained. Now $\frac{\delta y}{\delta x}$ approaches indefinitely close to $\frac{dy}{dx}$, the gradient of the tangent at P, while the right-hand side becomes in the limit nx^{n-1}.

Hence $$\frac{dy}{dx} = nx^{n-1}.$$

Example. *To find the gradient of the chord joining the points* $P \equiv (x, y)$ *and* $P_1 \equiv (x_1, y_1)$ *on the curve* $y = x^2$, *and to deduce the gradient of the tangent at the point* (x, y).

As before, we have $\ y = x^2\ $ and $\ y_1 = x_1^2$;

$$\therefore\ y_1 - y = x_1^2 - x^2 = (x_1 - x)(x_1 + x);$$

$$\therefore\ \frac{\delta y}{\delta x} \equiv \frac{y_1 - y}{x_1 - x} = x_1 + x.$$

Hence, putting $x_1=x$ and remembering that $\frac{dy}{dx}$ is the limiting value of $\frac{\delta y}{\delta x}$ when P_1 moves up to coincidence with P, we obtain $\frac{dy}{dx}=2x$.

The following table of results will be found convenient:

If $y=x$, $\therefore\ \frac{dy}{dx}=1$.

If $y=x^2$, $\therefore\ \frac{dy}{dx}=2x$.

If $y=x^3$, $\therefore\ \frac{dy}{dx}=3x^2$.

If $y=x^4$, $\therefore\ \frac{dy}{dx}=4x^3$.

If $y=x^5$, $\therefore\ \frac{dy}{dx}=5x^4$.

If $y=x^6$, $\therefore\ \frac{dy}{dx}=6x^5$.

……………………………

If $y=x^n$, $\therefore\ \frac{dy}{dx}=nx^{n-1}$.

7. Differential coefficient.

$\frac{dy}{dx}$ is also called the "**differential coefficient**" or "**derivative**" of y with respect to x.

8. Inclination of tangent to axis of x.

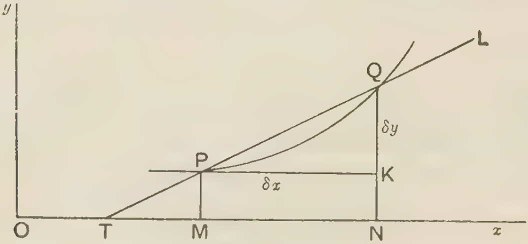


FIG. 19.

Plainly $\tan \mathrm{LT}x = \dfrac{\delta y}{\delta x}$.

Hence, if ψ denote the inclination of the tangent at P to Ox,

$$\therefore \tan \psi = \frac{dy}{dx}.$$

9. Rate of increase of y with respect to x.

We have seen in Chapter I. Art. 1, that the rate of increase of y with respect to x is given by the gradient of the tangent at the corresponding point on the graph representing the connexion between y and x.

Hence $\dfrac{dy}{dx}$, being the gradient of the tangent to the curve, represents the rate of increase of y with respect to x.

10. Further results :

(1) If $y = a$ (where a is a constant), $\therefore \dfrac{dy}{dx} = 0$.

(2) If $y = ax^n$, $\therefore \dfrac{dy}{dx} = nax^{n-1}$.

(3) If $y = ax^p + bx^q + cx^r$, $\therefore \dfrac{dy}{dx} = apx^{p-1} + bqx^{q-1} + crx^{r-1}$.

These results can easily be proved directly after the manner of Chap. II. Art. 4.

Example 1. *If* $y = 3 - 4x + 5x^2$, *find* $\dfrac{dy}{dx}$ *when* $x = 1$.

We have
$$\frac{dy}{dx} = -4 + 10x.$$
$$\frac{dy}{dx} = 6 \quad \text{when} \quad x = 1.$$

Example 2. *Find the angle of inclination of the tangent at* (1, 1) *on the curve* $y = \dfrac{x^2 - 2x + 4}{3}$ *to the axis of* x.

We have
$$\frac{dy}{dx} = \frac{2x - 2}{3}.$$
$$\therefore \frac{dy}{dx} = 0 \quad \text{when} \quad x = 1.$$

Hence the tangent at (1, 1) is parallel to the axis of x. This is also readily seen from the accompanying graph.

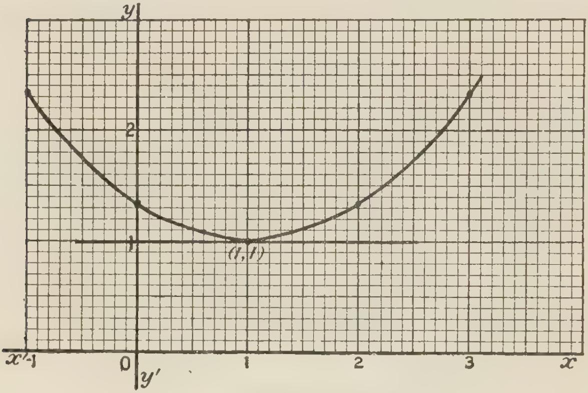


FIG. 20.

Example 3. *Calculate the angles at which the curve*

$$3y = x^2 - 3x - 10$$

cuts the axes.

We have $$\frac{dy}{dx} = \tfrac{2}{3}x - 1.$$

The curve cuts the axis of x where $x = -2$ and $+5$.

When $x = -2$, $\tan\psi \equiv \dfrac{dy}{dx} = -2{\cdot}3333$,

$$i.e.\ \psi = 113^\circ\ 12'.$$

When $x = 5$, $\tan\psi \equiv \dfrac{dy}{dx} = +2{\cdot}3333$,

$$i.e.\ \psi = 66^\circ\ 48'.$$

The curve cuts the axis of y where $x = 0$.

When $x = 0$, $\tan\psi \equiv \dfrac{dy}{dx} = -1$,

$$i.e.\ \psi = 135^\circ.$$

Example 4. *If $y = -7 + 4x - x^2$, find for what value of x, y has its greatest value, and calculate what that greatest value of* y *is.*

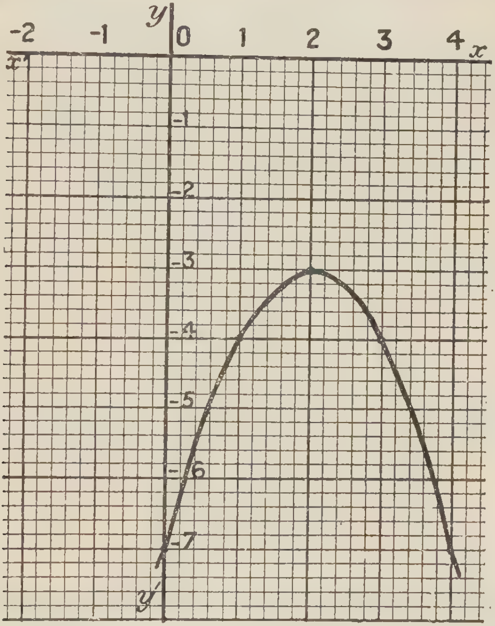


FIG. 21.

The graph shows that y has its greatest value when the tangent to the graph is parallel to the axis of x.

But $\frac{dy}{dx}$ is then zero.

$$\therefore\ 0 = \frac{dy}{dx} = 4 - 2x \quad \text{gives} \quad x = 2,$$

and the greatest value of y is -3.

11. Maxima and minima.

Definition. A point on a curve is said to be a **maximum (minimum)** if the value of y is greater (less) at that point than the values of y at neighbouring points.

The values of y at points representing maxima and minima are called "**turning values.**"

We shall now give examples of calculating maxima and minima in practical life by means of the Calculus.

We remember that if y is plotted against x, then at points on the graph at which y is either a maximum or a minimum the tangent is parallel to the axis of x, *i.e.* $\frac{dy}{dx}$ vanishes. A rough graph will show whether the points at which $\frac{dy}{dx}$ vanishes are maxima or minima.

Example 1. *Divide a straight line* 10 *cm. long into two parts such that the rectangle contained by the parts shall be a maximum.*

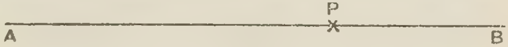


FIG. 22.

Let AB, 10 cm. long, be divided at P so that AP $=x$ cm. Then PB $=(10-x)$ cm.

Then A sq. cm. = area of rectangle whose sides are AP, BP

$$=x(10-x) \text{ sq. cm.}$$

$$\therefore \; A=-x^2+10x.$$

$$\frac{dA}{dx}=-2x+10.$$

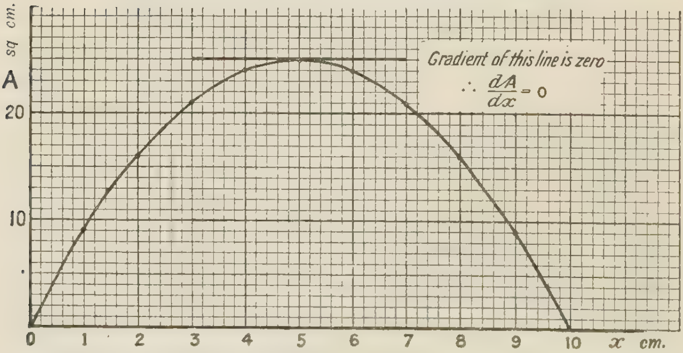


FIG. 23.

For a maximum the tangent is parallel to the axis of x on the (x, A) graph, *i.e.* $\frac{d\mathrm{A}}{dx}=0$. $\therefore\ x=5$, so that the line must be bisected. The accompanying graph illustrates this.

Example 2. *A man has four miles of fencing-wire and wishes to "fence in" a rectangular piece of prairie land through which a straight river flows, so as to be utilized as one side of the enclosure. How can he do this so as to enclose as much land as possible ?*

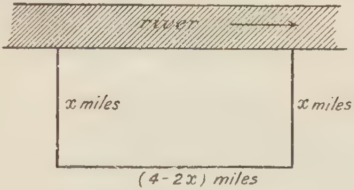


FIG. 24.

The figure shows how a rectangle can be fenced off from the prairie, utilizing the river as one side.

Then

$$\mathrm{A}\text{ sq. miles}=\text{area of the rectangle}$$
$$=x(4-2x)\text{ sq. miles.}$$
$$\therefore\ \mathrm{A}=4x-2x^2.$$
$$\therefore\ \frac{d\mathrm{A}}{dx}=4-4x.$$

If we plot A against x roughly, we see that the turning value of A is a maximum.

Hence, when A is as big as possible $\frac{d\mathrm{A}}{dx}=0$, and $\therefore\ x=1$, so the rectangle must have its side parallel to the river 2 miles long and the other sides 1 mile long.

Example 3. *A circular tin canister closed at both ends has a surface area of* 100 *sq. cm. Find the greatest volume it can contain.*

Let the radius of the circular end of the tin be r cm. and the height of the tin h cm. Let also the volume of the tin be V cub. cm.

Then

$$2\pi r^2+2\pi rh=100.$$
$$\mathrm{V}=\pi r^2h.$$

From the first equation $\pi rh=50-\pi r^2$.

$$\mathrm{V}=r\times\pi rh$$
$$=50r-\pi r^3.$$

Now, if we plot V against r, we shall see that the turning value for V is a maximum, and also that the tangent to the graph will be parallel to the axis of r, and consequently its gradient will be zero.

$$\therefore \frac{dV}{dr}=0 \quad \text{or} \quad 50-3\pi r^2=0.$$

$$\therefore r=\sqrt{\frac{50}{3\pi}}=2\cdot30 \text{ cm.}$$

$$\therefore V=r(50-\pi r^2)=r\left(50-\frac{50}{3}\right)$$

$$=\frac{100}{3}\sqrt{\frac{50}{3\pi}}=76\cdot78 \text{ c.c.}$$

EXAMPLES II.

1. Differentiate x^4, x^6, x^{12}, x^{17}, x^m, x^{2n}, x^{2n+1}.

2. Differentiate $3x^2$, $7x^9$, $8x^{14}$, $\frac{4}{3}x^2$, $\frac{3}{4}x^4$, $0\cdot5x^3$.

3. Differentiate $3+4x$, $3+4x^2$, $3x^2-4x^3$.

4. Differentiate $\frac{3}{4}x+\frac{4}{5}x^2$, $\frac{5}{4}x^2-\frac{7}{9}x^3$, $0\cdot8+0\cdot5x+0\cdot7x^2$.

5. If $y=3x^2-4$, calculate $\frac{dy}{dx}$ when $x=2$.

6. If $y=2x^2-4x+5$, calculate $\frac{dy}{dx}$ when $x=3$.

7. If $y=5x^3-3x$, calculate $\frac{dy}{dx}$ when $x=-1$.

8. If $y=2x^3-7x^2+3$, calculate $\frac{dy}{dx}$ when $x=-2$.

9. If $y=3x^2-4x+7$, find for what value of x, $\frac{dy}{dx}$ vanishes.

10. If $y=x^3-27$, find for what values of x, $\frac{dy}{dx}$ vanishes.

11. If $y=x^3-2x^2+x+5$, calculate those values of x for which $\frac{dy}{dx}$ vanishes.

12. If $y=0\cdot5x^2-0\cdot7$, calculate $\frac{dy}{dx}$ when $x=0\cdot9$.

13. If $y=0\cdot3x^2-0\cdot7x+0\cdot4$, calculate $\frac{dy}{dx}$ when $x=0\cdot5$.

14. If $y=x^3$, prove that $x\frac{dy}{dx}=3y$ for all values of x.

15. If $y=1-x^2$, prove that $x\frac{dy}{dx}+2=2y$ for all values of x.

16. If $y=x^2-3x+2$, find for what values of x, $\frac{dy}{dx}=y$. (Give your answer correct to two places of decimals.)

17. If $A=4\pi r^2$, calculate $\frac{dA}{dr}$ when $r=3$. (Give your answer correct to two places of decimals.)

18. If $V=\frac{4}{3}\pi r^3$, calculate $\frac{dV}{dr}$ when $r=4$. (Give your answer correct to two places of decimals.)

19. If $y=4x^2$ and $t=y\div\frac{dy}{dx}$, calculate t when $x=2{\cdot}5$.

20. If $s=4t^3$, $I=\left(s\times\frac{ds}{dt}\right)+\left(s\div\frac{ds}{dt}\right)$, calculate I when $t=0{\cdot}3$.

21. Sketch roughly $y=8x^2$ to see the nature of the curve; find $\frac{dy}{dx}$ in general, and hence calculate the values of $\frac{dy}{dx}$ when $x=0$, $0{\cdot}1$, $0{\cdot}5$, 2. Hence find from your tables the corresponding values of ψ.

22. Sketch roughly $y=4x^2-3x+2$ to get a general notion of its graph; find $\frac{dy}{dx}$, and hence calculate ψ when $x=-5$, $-0{\cdot}5$, $2{\cdot}7$, 3.

23. At what point on the graph of $y=1{\cdot}8x^2$ has ψ the value of 45° ?

24. At what point on the graph of $y=4x^2-3x+2$ has ψ the value 25° ?

25. In Qu. 24, find at what point on the graph the tangent is perpendicular to the tangent obtained in Qu. 24.

26. Draw roughly the graph of $y=4x^2-7x+3$. Then use Calculus to find accurately for what value of x the expression $4x^2-7x+3$ has its least value, and find what that least value is.

27. Draw roughly the graph of $y=34-17x-3x^2$. Then use Calculus to find accurately for what value of x the expression $34-17x-3x^2$ has its greatest value, and find what that greatest value is.

28. The line XY is 3 feet long. P is a point on XY distant x feet from X.

Calculate the area A of the rectangle whose sides are XP and PY in terms of x.

FIG. 25.

Plot A against x on squared paper and thus obtain a clear notion of the variation of the area of this rectangle as P moves from X to Y. Calculate by means of the Differential Calculus for what value of x, A is greatest, and find this value of A.

29. Find the gradient and inclination to the axis of x of the chord joining the points (2, 4) and (3, 9) on the curve $y=x^2$.

30. Find the gradient and inclination to the axis of x of the chord joining the points (1, 2) and (3, 10) on the curve $y=x^2+1$.

31. If (x, y) and $(x+\delta x, y+\delta y)$ are points on the curve $y=x^3$. prove that

$$\frac{\delta y}{\delta x}=3x^2+3x\,\delta x+(\delta x)^2.$$

Hence fill up the following table $\left(\text{where } \tan\phi=\frac{\delta y}{\delta x}\right)$:

x	1	1	1	1	1	1	1	1	1	1
δx	1	0·5	0·25	0·125	0·0625	0·2	0·02	0·002	0·0002	0·00013
$\frac{\delta y}{\delta x}$										
ϕ										

By the Calculus find $\frac{dy}{dx}$ and ψ $\left(\text{where } \tan\psi=\frac{dy}{dx}\right)$ at the point $x=1$, and compare with the approximate results obtained above.

32. If (x, y) and $(x+\delta x, y+\delta y)$ are points on the curve

$$y=0{\cdot}2x^2+0{\cdot}5x+0{\cdot}7,$$

calculate $\frac{\delta y}{\delta x}$ in terms of x and δx.

33. If (x, y) and $(x+\delta x, y+\delta y)$ are points on the curve $y^2=x^3$, prove that

$$2y\,\delta y+(\delta y)^2=3x^2\,\delta x+3x(\delta x)^2+(\delta x)^3.$$

34. A line AB 11 inches long is divided at P. Find P so that AP^2+PB^2 is a minimum.

35. A line AB 13 inches long is divided at P. Find P so that AP^3+PB^3 is a minimum.

36. What number exceeds its square by the greatest number possible ?

37. Fill in the vacant spaces in the following chart:

$$y = x^2$$

x	1	1	1	1	1	1	1	1	1	1	1
y											
δx	0·5	0·05	0·005	0·0005	0·00005	0·3	0·03	0·003	0·0003	0·00003	$\frac{1}{4987}$
δy											
$\frac{\delta y}{\delta x}$											

What is the value of $\frac{dy}{dx}$ when $x = 1$?

What do you deduce from the above investigation?

38. What number exceeds its cube by the greatest number possible?

39. A piece of wire 5 feet long is cut into two parts. One part is bent to form a square and the other part to form a circle. What are the lengths of the parts when the sum of the areas formed is a minimum?

40. If $\frac{d^2y}{dx^2}$ be defined as the differential of $\frac{dy}{dx}$, *i.e.* $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, find $\frac{d^2y}{dx^2}$ if $y = 2 + 3x + 5x^2 + 6x^3$.

41. If $y = 0·2 - 0·3x - 0·5x^2 + 0·7x^3$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

42. If $y = 0·7 - 0·4x^4$, calculate $\frac{d^2y}{dx^2}$ when $x = 0$ and when $x = 1$.

43. If $y = 0·51x^2 - 0·23x$, calculate $\frac{d^2y}{dx^2}$ when $x = -1$.

44. If $z = 3y + 5\frac{dy}{dx} - 4\frac{d^2y}{dx^2}$, calculate z when $x = 1$, given that $y = x^3 + x^4$.

45. Find the gradient of the tangent to the curve $y = 3x^2 - 4x + 5$ at the point (1, 4).

46. Find the gradient of the tangent to the curve

$$y = 8x^7 - 1·5x^4 + 0·7x$$

at the point (0, 0).

47. Find the gradient of the tangent to the curve $y=0{\cdot}2x-1{\cdot}3x^4$ at the points (0, 0) and (1, $-1{\cdot}1$).

48. Express x^2, x^3, x^4, $5x^5$, x^2+1, $2x^2-3$, $3x^2-2x+4$, $4x^3+5x-9$, $7-4x$, as differential coefficients of other functions. (See Art. 7, Chap. II.)

49. Draw the graph of $3y=x^2+x-6$. (*a*) Find the points in which the curve cuts the axis of x, and calculate the inclinations to $x'x$ of the tangents at these points. (*b*) Find the point in which the curve cuts the axis of y, and find the inclination to $x'x$ of the tangent at that point.

50. A flexible ruler 2 feet long is fixed horizontally into a wall at O, and a light weight is hung on at A. It can be proved that a point on the ruler distant x inches from O bends through y inches, given by the formula

$$y=\frac{1}{24^3}(72x^2-x^3).$$

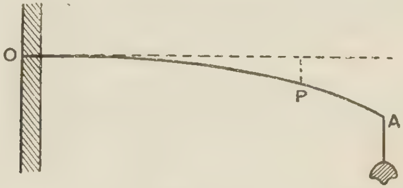


FIG. 26.

(*a*) Calculate the deflexion of points distant 4 inches, 6 inches, 12 inches, 15 inches, 24 inches from O.

(*b*) Calculate the curvature at the same points, given that the curvature is measured by $\frac{d^2y}{dx^2}$.

51. If $y=2x+\frac{1{\cdot}5}{x}$, state what value of x will make y less than any other. (Bd. of Educ.)

52. Sheet lead of unlimited length and width 4 feet has to be bent into the form of a pipe with a cross-section ⊔ (open at the top) to act as a duct for water. What is the best way to bend it so as to let as much water through as possible? If water is flowing in the pipe at the rate of one foot per second, find the volume of water discharged from the duct per second, assuming that the pipe is full.

53. A man orders a plumber to make a cistern with a square base and closed at the top to hold 1000 cu. ft. of water. It has to be lined inside with sheet lead at $4\frac{1}{2}$d. per square foot. Find the least that the cost of lining can be.

54. Solve the last question when the cistern is open at the top.

55. A tin canister with a square base is tied with 40 inches of string (neglecting knots). The string passes across the centre of the lid parallel to the side, down one side of the box, across the base and up the other side. Find the greatest volume that can thus be tied up with this piece of string.

56. A piece of wire 3 feet long is cut into two parts. One part is bent to form a square and the other part to form an equilateral triangle. What are the lengths of the parts when the sum of the areas formed is a minimum ?

57. A piece of tinplate 20 inches square has squares cut off each of the 4 corners, and the ends are then turned up to form an open box. Find the dimensions of the box obtained, assuming that the squares cut out have each a side of x inches. Find also the volume of the box in terms of x.

What is the maximum volume of this box ?

58. A rectangular piece of tinplate 29 inches by 24 inches has squares cut off each of its 4 corners, and the ends are then turned up to form an open biscuit tin. Find the side of the square so that the biscuit tin may have a maximum volume, also so that it may have a minimum volume.

59. Post Office regulations prescribe that the combined length and girth of a parcel must not exceed 6 feet. Find the maximum volume of a parcel whose shape is a prism with the ends squares.

60. Find the maximum volume of a parcel whose shape is a circular cylinder.

61. CAB is an isosceles triangle, right-angled at C and having CA = CB. P lies on AB, and PM is perpendicular to CA and PN is perpendicular to CB. Find the position of P in which the area of the rectangle PMCN is greatest.

CHAPTER III.

APPLICATION OF THE CALCULUS TO RATES, VELOCITIES ETC.

1. Uniform motion.

We have already seen in Chap. I. Art. 1, that if a body be moving with uniform velocity in a straight line, we may plot the distance s traversed against the time t, in which case we obtain as the (t, s) graph a straight line. We have also seen that the gradient of this straight line represents the velocity of motion.

Thus, on consulting a time-table, we find that a certain non-stopping train leaves Bristol at 12 noon and reaches London at 2 p.m., the distance traversed being $118\frac{1}{4}$ miles. If the velocity of the train be conceived absolutely uniform, the graph obtained on plotting the distance against the time will be the line joining the origin to the point $(120, 118\frac{1}{4})$, the *time* being measured in minutes and the *distance* in miles. Having drawn the graph, we can "read off" very approximately the times at which the train passes the various stations thus:

Distance from Bristol	Badminton, 19 miles	Didcot, $64\frac{1}{2}$ miles	Reading, $81\frac{1}{2}$ miles	Maidenhead, $94\frac{1}{4}$ miles
Times of passing	12.19	1.5	1.23	1.36

The gradient of the graph is, of course, $\frac{118\frac{1}{4}}{120}$ miles per min., which can be "read off" more conveniently in the form 0·98 ml. per 1 min.

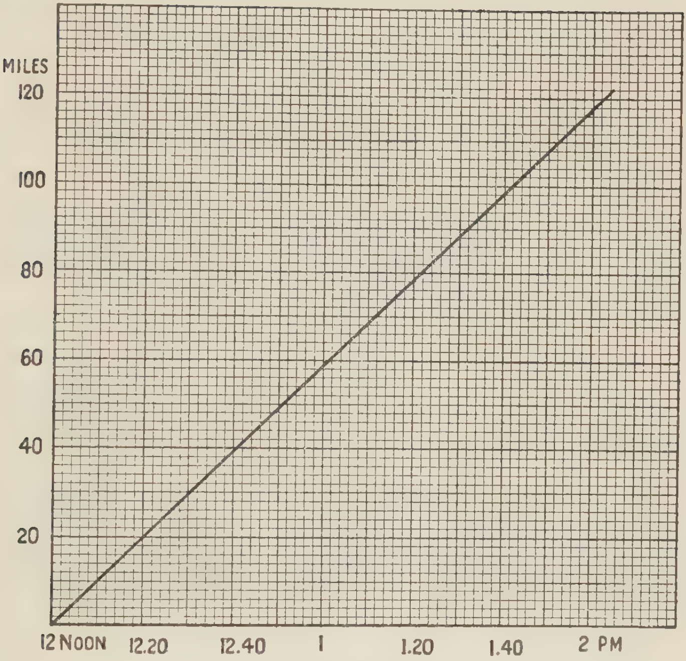


FIG. 27.

2. Variable velocity.

In general, in Nature, particles, trains, etc., moving in straight lines, do not go with uniform velocity. Hence the result of plotting the distance against the time will not in general be a straight line. To obtain the velocity from the "*time-distance*" graph, we draw the tangent to the graph at the point under investigation. The gradient of this tangent represents the velocity required. (See Chap. I. Art. 1.)

Now if s cm. be the distance and t sec. be the time, and if we plot s against t, the gradient of the tangent will be given by $\frac{ds}{dt}$, with the proper units attached.

Hence, if s be measured in cm. and t in sec., the velocity is measured by $\frac{ds}{dt}$ cm. per sec.

Again, if P denote the population of a town and t the time in years, the rate at which the population is growing will be measured by $\frac{dP}{dt}$ people per annum.

If water be running into a trough and if V cu. ft. be the volume of water in the trough and t hours the time during which the tap has been turned on, the rate at which the trough is filling is $\frac{dV}{dt}$ cubic feet per hour.

In a certain pot the water is x feet deep and the volume of the water is V cubic feet, the rate at which the volume of the water is increasing relative to the depth of water in the pot is $\frac{dV}{dx}$ cubic feet per foot.

When a ship is being loaded, she sinks deeper and deeper into the water the more that is taken aboard. If x be the number of feet of water that she draws when there is a load of L tons on board, the rate at which she draws water is given by $\frac{dx}{dL}$ feet per ton of cargo.

In general, if y be compared with x, the rate at which y is increasing relative to x is $\frac{dy}{dx}$.

Example 1. *A particle is moving along a straight line, and is at a distance of s cm. from a fixed point on the line, t sec. after passing that point. If the motion is given by* $s = 7t + 11t^2$, *find the velocity* (1) *t sec.*, (2) 2 *sec. after the beginning of the motion.*

$$\because \quad s = 7t + 11t^2,$$

$$\therefore \quad \frac{ds}{dt} = 7 + 22t.$$

Hence the velocity after t sec. is $(7 + 22t)$ feet per second, and the velocity after 2 seconds is 51 ft. per sec.

Example 2. *In a certain machine, the effort* P *lb. applied by the hand holds up a load of* W *lb., where* P *and* W *are connected by the* *formula* $P = 0{\cdot}41W + 79$. *Find the rate at which the effort* P *has to be increased as the load* W *gets heavier.*

We have to find $\frac{dP}{dW}$ lb. per lb.

But $\frac{dP}{dW} = 0{\cdot}41$ from the given formula.

$\therefore$ P is increasing relatively to W at the rate of 0·41 lb. per lb. increase of load.

Example 3. *A conical vessel contains* V *c.c. of liquid when the depth of liquid is x cm., where $V = 0{\cdot}83x^3$. Find the rate at which the volume increases compared with the depth when the liquid is* 2 *cm. deep.*

We have to find $\frac{dV}{dx}$ cu. cm. per cm.

Now $\frac{dV}{dx} = 2{\cdot}49x^2$.

$\therefore$ the volume increases at the rate of $2{\cdot}49x^2$ cu. cm. per cm.

Putting $x = 2$, we obtain that the volume increases at the rate of 9·96 cu. cm. per cm. when the depth is 2 cm.

Example 4. *Water is pouring steadily from a spout at the rate of a cubic foot per minute into a rectangular trough whose base is* 12 *sq. inches. Find the rate at which the water is rising in the trough.*

If V cubic inches be the volume of water in the trough when the depth of water is x inches, then

$$V = 12x.$$

$$\therefore \frac{dV}{dt} = 12\frac{dx}{dt}. \quad \text{..(1)}$$

But

$$\frac{dV}{dt} = 1728 \text{ (cubic inches per min.)}. \text{......(2)}$$

$$\therefore 1728 = 12\frac{dx}{dt}, \text{ by (1) and (2)}.$$

$$\therefore \frac{dx}{dt} = 144.$$

Hence the water is rising at the rate of 144 inches per minute.

Example 5. *An electric car is going at* 15 *ml. per hr. The overhead wheel that presses on the conducting wire is* 6 *inches in diameter. Find the rate at which it revolves.*

Let x ft. be the distance of the car from some fixed point,—say the tramway-station.

Let the overhead wheel have revolved through θ radians while the car was traversing x feet.

Hence (since arc = radius × angle subtended at centre),

$$x = \tfrac{1}{4}\theta.$$

$$\therefore \quad \frac{dx}{dt} = \frac{1}{4}\frac{d\theta}{dt}. \quad \text{..................................(1)}$$

But $\quad \dfrac{dx}{dt} = 22$ ft. per sec. ($\because$ 15 m.p.h. = 22 f.p.s.). ....(2)

Hence $\quad \dfrac{d\theta}{dt} = 88$ radians per sec., by (1) and (2),

$$= \frac{88 \times 180}{\pi} \text{ degrees per sec.}$$

$$= 5041 \text{ degrees per sec. (approx.)}.$$

3. Acceleration.

The rate at which the velocity of a body varies plays a very important part in Mechanics.

The acceleration is the rate at which the velocity of a body is changing.

Hence if v be the velocity of the body at the time t, the acceleration at that instant will be measured by $\dfrac{dv}{dt}$.

The acceleration will be represented graphically on the time-velocity diagram by the gradient with respect to the time-axis of the tangent to the graph at the particular instant under consideration.

Note.—If the "distance" (s) be measured in (feet) and "time" (t) in (seconds), then the "velocity" $\left(\dfrac{ds}{dt}\right)$ is measured in (feet) per (second).

Similarly, if the "velocity" (v) be measured in (feet per second) and the "time" (t) in (seconds), then the "acceleration" $\left(\dfrac{dv}{dt}\right)$ is measured in (feet per second) per (second).

The contents of the various brackets are merely interchanged.

Example 1. *A brick falls from a chimney to the ground. It is known experimentally that s the distance fallen, measured in feet, is connected with the time t taken, measured in seconds, by means of the formula $s = 16t^2$. Find the acceleration at the instant t.*

We have $s=16t^2$.

$$\therefore \frac{ds}{dt}=32t.$$

$$\therefore v=32t.$$

$$\therefore \frac{dv}{dt}=32.$$

Hence the acceleration is 32 ft. per sec. per sec. In other words, a falling body acquires an increase of velocity at the constant rate of 32 feet per sec. every second.

Example 2. *Two equal weights* W *and* W *are attached to the ends of a string passing over two smooth pegs* A *and* B *close together and supporting the small weight* w *as shown. Calculate the ratio of the velocities and the accelerations of the* W's *and* w *as the* W's *descend and pull* w *up.*

Let x feet and y feet be the respective distances of w and W below the pegs.

Let the string be 8 feet long.

Hence $2y+2x=8$.

$$\therefore y+x=4.$$

$$\therefore y=4-x.$$

$$\therefore \frac{dy}{dt}=-\frac{dx}{dt}.$$

Hence if u ft. per sec. be the velocity of ascent of w, and v ft. per sec. the velocity of descent of the W's,

$$\therefore v=u, \quad\text{..............................(1)}$$

i.e. the velocities of the W's and w are always equal and opposite.

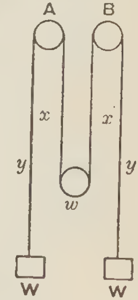


FIG. 28.

Again, from (1), $\frac{dv}{dt}=\frac{du}{dt}$ ....(2)

Hence the accelerations of the W's and the w are always equal and opposite.

4. Second differentials.

We have seen that the velocity $v=\frac{ds}{dt}$,

and that the acceleration $f=\frac{dv}{dt}$.

$$\therefore f=\frac{d}{dt}\left(\frac{ds}{dt}\right).$$

This is more usually written $\frac{d^2s}{dt^2}$ and is called the "second differential of s with respect to t." $\frac{d^2s}{dt^2}$ is read "d two s by dt square."

Second differentials play a very important part in Mechanics, Radii of Curvature, etc.

If a body be moving in a straight line, so that the body is at a distance s ft. measured from a fixed point on the line after t sec., then the acceleration is measured by $\frac{d^2s}{dt^2}$ *in feet per sec. per sec.*

Example 1. *If* $y=3x^3-4x^2+5x+7$, *find* $\frac{d^2y}{dx^2}$.

We have
$$\frac{dy}{dx}=9x^2-8x+5.$$
$$\therefore \frac{d^2y}{dx^2}=18x-8.$$

Example 2. *A body moves in a straight line so that it describes s feet in time t sec. according to the law* $s=7{\cdot}3t+4{\cdot}6t^2$. *Prove that the acceleration is constant, and find its value.*

We have
$$\frac{ds}{dt}=7{\cdot}3+9{\cdot}2t.$$
$$\therefore \frac{d^2s}{dt^2}=9{\cdot}2.$$

Hence the body moves with a constant acceleration of 9·2 ft. per sec. per sec.

EXAMPLES III.

1. A body falls from a tower according to the law $s=16t^2$, where s is measured in feet and t in seconds. Find the velocity after 1, 2, 3, 4, 5 seconds respectively.

2. A body slides down a smooth inclined plane according to the law $s=2{\cdot}3t+4{\cdot}7t^2$, where s is measured in cm. and t in sec. Find the velocity after 1, 3, 5 seconds respectively.

Prove that the acceleration is constant, and find its value.

3. Heat is being communicated continuously to a body so that the temperature at time t minutes from the original observation is T degrees Centigrade. Express in the notation of the Differential Calculus the rate of rise of temperature, and give the proper units.

4. The barometric pressure in cm. of mercury is p at a height h feet up a mountain. Express the rate of rise of pressure as one ascends the mountain in the symbolism of the Calculus, and give the proper units.

5. Gas is contained in a heat-proof case. The pressure is p lb. per sq. in. and the temperature is T degrees Centigrade. Heat is introduced so that the pressure and temperature gradually rise. Express in the notation of the Calculus the rate of increase of the pressure per unit rise in the temperature.

6. Gas is contained in a cylinder and is maintained at the same temperature. There is a moveable piston in the cylinder which regulates the volume. If the pressure is p lb. per square in. when the volume is v cubic feet, find the ratio of the rates at which the pressure and volume alter as the piston is being pushed in.

7. When the wind is blowing at the rate of v miles per hour, the pressure on a signboard is P lb. per sq. ft. Find the relative rate at which the pressure on the board and the velocity of the wind increase and give the proper units.

8. In a certain machine, the effort exerted by the operator is connected with the weight W raised by means of the formula $E = 0{\cdot}3W + 20$, E being measured in lb.-wt. and W in cwt. Find the relative rate of increase of E and W as the weight to be raised is increased. (Be careful to give the proper units.)

9. Find the relative rate at which the side of a square and its area are increasing, the sides being x feet.

10. Find the ratio of the rates at which the side of a cube and its volume are increasing, the side being x cm.

11. A stone is projected upwards with a velocity of 80 ft. per sec. and the height s feet is known to be connected with the time of rise t sec. by means of the formula $s = 80t - 16t^2$.

(i) Find the velocity after t sec.
(ii) Find the velocity after 1, 2, 3, 4, 5 seconds.
(iii) Find the acceleration.
(iv) Find when the stone reaches its highest point.
(v) Find the velocity when the stone is half-way up, (b) half-way down.

12. A stone is projected upwards with a velocity of 100 ft. per sec. Its equation of motion is known to be $s = 100t - 16t^2$.

(1) Find when it is moving with a velocity of 80 ft. per sec.
(2) Find when it reaches its highest point.
(3) Find how far up the stone goes before it begins to descend.

13. A gun shoots upward at a Zeppelin, and the shot is discharged with a velocity of 2440 ft. per sec. Assuming that the shot moves upward according to the law $s = 2440t - 16t^2$, find how far up the Zeppelin must be before it is out of range.

14. A wheel is spinning according to the law $\theta = 47 + 15t - 2t^2$, where θ is measured in degrees and t in seconds.

(i) Find the angular velocity of the wheel at time t, in degrees per second.

(ii) Find the angular velocity of the wheel after 1, 2, 3 sec. respectively.

(iii) Find the initial angular velocity of the wheel.

(iv) Prove that the wheel spins with uniform retardation.

(v) Find after how long the wheel ceases to spin.

15. A train runs s miles in t hours, where $s = 50t - 6t^2$. What is the speed of the train after half an hour?

16. A bicycle with wheels 28″ in diameter freewheels down a slope so that it covers s feet in t seconds, where $s = \frac{8}{25}t^2$. Find how fast the wheel is revolving after half a minute.

17. P is a point on Oy and N is the foot of the perpendicular dropped from P to Ox. If x and y be the respective distances of N and P from O, express x in terms of y.

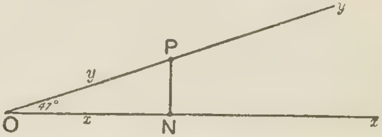


FIG. 29.

Hence find the velocity of N in terms of the velocity of P.

If P be moving at the rate of 3 cm. per sec., calculate the velocity of N.

If N be moving at the rate of 8 ft. per sec., calculate the velocity of P.

18. Two trains approaching a station on parallel lines seem to their passengers to be going at the same rate. One is going at 40 miles per hour on the level and the other is running down a gradient of 1 in 7. What is the speed of the latter train?

19. Two squirrels are in a tree at the same vertical distance from the ground, one on the main trunk and the other on a branch making with the trunk an angle of 70°. The squirrel on the main trunk descends at the rate of 6 ft. per sec. and meets its friend at the fork of the branch. What is the speed of the other squirrel ?

20. A string 8 ft. long is attached to the point A. B is a smooth pulley. The small weight w is slung on the string by means of a smooth ring, and a heavy weight W is attached at the free end.

If x ft. be the depth of w and y ft. that of W below the ceiling, find y in terms of x.

Hence find the velocity with which W descends relative to that with which w ascends.

Also find the acceleration with which W descends relative to that with which w ascends.

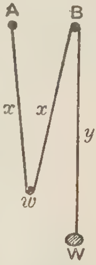


FIG. 30.

21. P is a fixed pulley. X is a revolving drum 12 inches in diameter. A wire-rope passes round this drum, over P, and is attached to a weight W. If the drum revolves through θ degrees, and W rises through x inches, find the algebraic connexion between x and θ. Hence deduce the algebraic connexion between

$$\frac{dx}{dt} \text{ and } \frac{d\theta}{dt}.$$

If the drum goes at 4 revolutions per second, find how fast W is rising.

22. Two tanks with square bases are placed one above the other, and with a pipe connecting them. The sides of the bases of the tanks are 2 feet and 6 feet respectively. 12 cubic feet of water is poured into the upper, and the water then passes into the lower. If at any instant there be y feet of water in the upper and x feet in the lower, find the algebraic connexion between x and y (neglecting the volume of the connecting pipe).

Find the ratio of the rates at which the levels of water are rising and falling in the respective tanks.

23. In Question 22, find the ratio of the respective rates of rising and falling if the lower tank has a square base of side 8 feet and the upper a circular base of diameter 3 feet.

24. Water is pouring steadily from a tap into a bath at the rate of 3 cu. ft. per min. The bath has a rectangular base of 6 feet by $2\frac{1}{2}$ feet. If V cu. ft. be the volume of water in the bath and x ft. the depth of water, find the connexion between V and x. Hence deduce the connexion between $\frac{dV}{dt}$ and $\frac{dx}{dt}$. At what rate is the water rising in the bath ?

25. Water is pouring steadily from a tap into a circular tub at the rate of 2 cu. ft. per min. If the base of the tub has a diameter of 3 feet, find the rate at which the water is rising in the tub.

26. Water is pouring steadily from a tap into a bath at the rate of 3 cu. ft. per min. The bath has a rectangular base of 15 sq. feet and vertical sides, and owing to the waste-pipe being open the water rises at the rate of 1 inch per min. Find the rate of discharge of the waste-pipe.

CHAPTER IV.

FURTHER DISCUSSION OF RATES: GEOMETRY OF CURVES.

1. Useful result, viz. $\frac{dz}{dx}=\frac{dz}{dy}\cdot\frac{dy}{dx}$.

If y be expressed as a function of x and z as a function of y, it is often useful to find the rate at which z is varying with respect to x, even if z be not expressed in terms of x.

We have, with the usual notation,

$$\frac{\delta z}{\delta x}=\frac{\delta z}{\delta y}\cdot\frac{\delta y}{\delta x}$$

by cancelling δy. Hence, proceeding to the limit,

$$\frac{dz}{dx}=\frac{dz}{dy}\cdot\frac{dy}{dx}.$$

Example 1. *If* $z=5y+7$ *and* $y=3x-4$, *to find* $\frac{dz}{dx}$, *where* z *is not expressed in terms of* x.

Plainly $$\frac{dz}{dy}=5 \quad\text{and}\quad \frac{dy}{dx}=3.$$

$$\therefore\ \frac{dz}{dx}=\frac{dz}{dy}\cdot\frac{dy}{dx}=5\times 3=15.$$

Verification.

We have $z=5y+7$

$$=5(3x-4)+7 \text{ (by substituting for } y \text{ in terms of } x)$$

$$=15x-13.$$

$$\therefore\ \frac{dz}{dx}=15.$$

Example 2. *If* $z=3y^2+4$ *and* $y=2x-1$, *to find* $\frac{dz}{dx}$, *where* z *is not expressed in terms of* x.

Plainly $$\frac{dz}{dy}=6y \quad \text{and} \quad \frac{dy}{dx}=2.$$

$$\therefore \frac{dz}{dx}=\frac{dz}{dy}\times\frac{dy}{dx}=6y\times 2=12y.$$

Verification.

We have $$\begin{aligned} z&=3y^2+4 \\ &=3(2x-1)^2+4 \text{ (by substituting for } y \text{ in terms of } x) \\ &=12x^2-12x+7. \end{aligned}$$

$$\therefore \frac{dz}{dx}=24x-12=12(2x-1),$$

which is the same as the result obtained above, since $y=2x-1$.

We shall now give some practical applications. Example 3 exhibits a method of great and universal utility, and is developed at greater length in Chap. V. Art. 5, under the title of "*Method of Substitution.*"

Example 3. *If* $z=(1+x)^{12}$, *to find* $\frac{dz}{dx}$.

Let $1+x=y$. $\quad\therefore \frac{dy}{dx}=1.$ ....(1)

Also $z=(1+x)^{12}=y^{12}$. $\quad\therefore \frac{dz}{dy}=12y^{11}.$ ....(2)

$$\begin{aligned} \therefore \frac{dz}{dx}=\frac{dz}{dy}\times\frac{dy}{dx}&=12y^{11}\times 1 \\ &=12y^{11} \\ &=12(1+x)^{11}. \end{aligned}$$

Example 4. *A circular metal plate is being heated, and its radius at a given instant is 3 inches, and is increasing at the rate of* $\frac{1}{100}$*th of an inch per hour. Calculate the rate at which its area is increasing.*

Let A sq. in. be the area when the radius is r in.

$$\therefore \mathrm{A}=\pi r^2.$$

$$\begin{aligned} \therefore \frac{d\mathrm{A}}{dt}&=\pi\frac{dr^2}{dt} \\ &=\pi\frac{dr^2}{dr}\times\frac{dr}{dt} \\ &=2\pi r\times\frac{dr}{dt}. \quad\text{..................................(1)} \end{aligned}$$

But $$\frac{dr}{dt}=\tfrac{1}{100} \text{ (inch per hour).} \qquad\qquad (2)$$

$$\therefore \frac{dA}{dt}=2\pi\times 3\times\tfrac{1}{100} \text{ sq. in. per hour}$$

$$=\frac{3\pi}{50} \text{ sq. in. per hour}$$

$$=0{\cdot}19 \text{ sq. in. per hour.}$$

Otherwise.

Since $$A=\pi r^2,$$

$$\therefore \frac{dA}{dr}=2\pi r$$

$$=6\pi \text{ (when } r=3).$$

But A is increasing $\frac{dA}{dr}$ times as fast as the radius, *i.e.* A is increasing 6π times as fast as the radius. Now the radius is increasing at the rate of $\frac{1}{100}$th inch per hour. Hence A is increasing at the rate of $6\pi\times\frac{1}{100}$ sq. in. per hr.,

$$i.e.\ \frac{3\pi}{50} \text{ sq. in. per hr.,}$$

$$i.e.\ 0{\cdot}19 \text{ sq. in. per hr.}$$

Example 5. *Water is pouring steadily from a spout at the rate of one cubic foot per minute into a conical reservoir whose semi-angle at the apex is 30°. Find the rate at which the depth of water is increasing.*

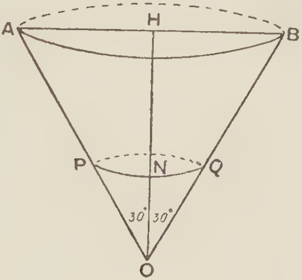


FIG. 31.

Let $$ON=x \text{ inches.}$$

Let $$V \text{ cu. in.}=\text{volume of } POQ.$$

Hence $$\text{PN} = \text{ON} \tan 30^\circ = \frac{x}{\sqrt{3}}.$$

$$\therefore \quad \text{V} = \tfrac{1}{3} \times \pi \frac{x^2}{3} \times x = \frac{\pi x^3}{9}.$$

$$\therefore \quad \frac{d\text{V}}{dt} = \frac{\pi}{9} \frac{dx^3}{dt}$$

$$= \frac{\pi}{9} \times \frac{dx^3}{dx} \times \frac{dx}{dt}$$

$$= \frac{\pi}{3} x^2 \frac{dx}{dt}. \quad \text{.....................................(1)}$$

But $$\frac{d\text{V}}{dt} = 1 \text{ cubic foot per min. ....(2)}$$

$$\therefore \quad \frac{dx}{dt} = \frac{3}{\pi x^2} \text{ feet per min.}$$

Hence when x is very small, the water rises very rapidly in the reservoir, but as it gets fuller the depth increases more and more slowly.

We are familiar with this phenomenon, and the above is an example of how calculations can be used to explain and measure accurately events in everyday life.

2. The student being now familiar with the simpler processes and interpretations of differentiation, we shall give some examples of the use of these processes to find the equations of tangents to curves.

3. *To find the equation to a straight line passing through the point (h, k) and inclined at an angle ψ to the axis of x.*

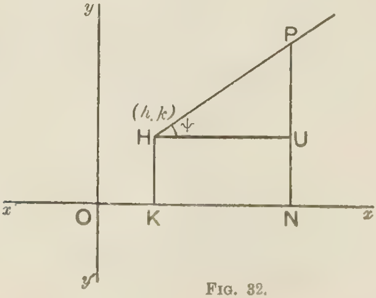


FIG. 32.

Let HL be the straight line passing through the point $H \equiv (h, k)$ and inclined at an angle ψ to the axis of x.

Let $P \equiv (x, y)$ be a variable point on the line HL.

Draw HK and PN perpendicular to $x'x$ and HU parallel to $x'x$.

Then $$\frac{UP}{HU} = \tan\psi.$$

But $$UP = NP - NU = y - k,$$
and $$HU = ON - OK = x - h.$$
$$\therefore \frac{y-k}{x-h} = \tan\psi,$$
i.e. $(y-k) = (x-h)\tan\psi$ is the equation required to the line HL.

Example. *Find the equation to the straight line passing through the point* (2, 3) *and inclined at an angle of* 45° *to the axis of* x.

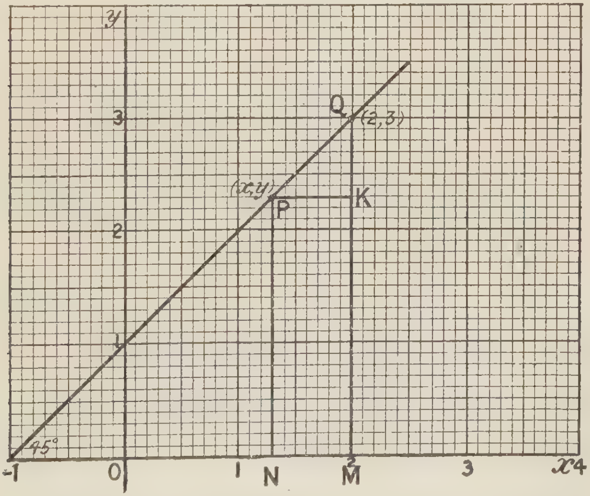


FIG. 33.

We have $$KQ = MQ - NP = 3 - y,$$
$$PK = OM - ON = 2 - x.$$

But $$\frac{KQ}{PK}=\tan 45^\circ=1.$$

$$\therefore\ 3-y=2-x,$$

or $x-y+1=0$ is the equation to the line PQ.

4. Application to curves.

If (h, k) be a point on a curve, to find the equation of the tangent to the curve at that point.

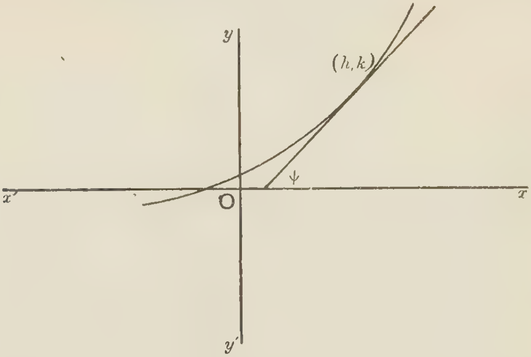


FIG. 34.

As in the last article, the equation of the tangent is

$$\frac{y-k}{x-h}=\tan\psi,$$

$$i.e.\ (y-k)=(x-h)\tan\psi,$$

where $\tan\psi$ is the value of $\dfrac{dy}{dx}$ at the point (h, k).

An example will make the method of procedure plain :

Example. *To find the equation of the tangent to the curve* $y=5x^2-2x+3$ *at the point* (1, 6).

It is easy to verify that the point (1, 6) lies on the curve.

Also $$\frac{dy}{dx}=10x-2$$
$$=8\ (\text{at the point } (1, 6)\,).$$

Hence the equation of the tangent at (1, 6) is

$$\frac{y-6}{x-1}=8,$$

$$\text{i.e. } y-6=8(x-1),$$

$$\text{i.e. } y=8x-2.$$

N.B.—It is easy to verify by direct substitution that the line $y=8x-2$ cuts the curve $y=5x^2-2x+3$ in *two* coincident points, since $8x-2=5x^2-2x+3$ gives $5(x^2-2x+1)=0$, *i.e.* $(x-1)^2=0$.

5. The following definitions are required in studying the geometrical theory of curves :

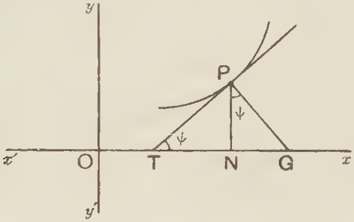


FIG. 35.

Let P be a point on a given curve and let PN be the perpendicular from P to the axis of x.

Let the tangent at P meet $x'x$ in T.

Let PG be perpendicular to PT at P.

Then PG is called the "**normal**" to the curve at P.

TN is called the "**subtangent**" to the curve at P.

NG is called the "**subnormal**" to the curve at P.

6. Curvature.

We are already familiar with the "tangent" to a given curve at an assigned point P on the curve, and with the notion that for certain purposes we can regard a small portion of the arc of a curve as approximately a straight line. For other and more accurate purposes, we must regard a small portion of the arc of a given curve as approximately the arc of a circle. We require to do so in studying the motion of a train round a curved portion of the railway-line or the motion of a car on a switch-back railway.

We have already seen how we find the tangent, but to find a formula for the "radius of curvature" (as the radius of the above circle is called) at a given point of a curve requires too hard mathematics for the present stage of the student. We will therefore content ourselves with saying:

(1) The "radius of curvature" at any point of a curve is usually denoted by ρ.

(2) The formula for calculating ρ is

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

(3) $\frac{1}{\rho}$ (*i.e.* the reciprocal of ρ) is called "the curvature" of the curve at any assigned point.

Example 1. *Calculate the subtangent at the point* (4, 2) *of the parabola* $8y = x^2$.

We have $$8\frac{dy}{dx} = 2x.$$

$$\therefore \frac{dy}{dx} = \frac{x}{4}.$$

$$\therefore \tan\psi = \frac{dy}{dx} = 1 \text{ at the point } (4, 2).$$

Now referring to the figure in Art. 5,

$$\frac{\text{NP}}{\text{TN}} = \tan\psi = 1.$$

$$\begin{aligned} \therefore \text{ Subtangent} &\equiv \text{TN} \\ &= \text{NP} \\ &= y \text{ of the point } (4, 2) \\ &= 2. \end{aligned}$$

Example 2. *Prove that the subnormal has always the same value for all points of the parabola* $y^2 = 4ax$, *and find its length.*

We have $$\tan\psi = \frac{dy}{dx}.$$

But $$x = \frac{y^2}{4a}. \quad \therefore \frac{dx}{dy} = \frac{y}{2a}. \quad \therefore \tan\psi = \frac{2a}{y}.$$

$$\begin{aligned} \text{Now, the subnormal} &\equiv \text{NG} \\ &= \text{NP} \times \tan\psi, \text{ referring to Art. 5,} \\ &= y \times \frac{2a}{y} \\ &= 2a. \end{aligned}$$

That is, the subnormal has the same value whatever point (x, y) we take on the curve.

Example 3. *Calculate the radius of curvature at the point* (3, 6) *of the curve* $y = x^2 - x$ *and also at the origin.*

We have
$$\frac{dy}{dx} = 2x - 1,$$
and
$$\frac{d^2y}{dx^2} = 2.$$
$$\therefore\ \rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$
$$= \frac{\{1 + 4x^2 - 4x + 1\}^{\frac{3}{2}}}{2}$$
$$= (2x^2 - 2x + 1)^{\frac{3}{2}}\sqrt{2}.$$

$\therefore$ radius of curvature at $(3, 6) = (13)^{\frac{3}{2}}\sqrt{2}$,

and radius of curvature at the origin $= \sqrt{2}$.

EXAMPLES IV.

1. If $y = 1 + x + x^2$, find $\frac{dy}{dt}$ in terms of x and $\frac{dx}{dt}$.

2. In Question 1, if $x = 2$ cm. and $\frac{dx}{dt} = 3$ cm. per sec., calculate $\frac{dy}{dt}$.

3. If $y = 2x^4 - 5x^2 + 3x - 7$, calculate $\frac{dy}{dt}$ in terms of x and $\frac{dx}{dt}$.

4. If $V = x^3$, find $\frac{dV}{dt}$ in terms of x and $\frac{dx}{dt}$.

5. A metal cube is being heated so that the side is increasing at the rate of 0·02 cm. per hr. Calculate the rate at which the volume is increasing when the side is 5 cm.

6. The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$. An elastic spherical balloon is being blown up so that the radius is increasing at the rate of 1 cm. per sec.; calculate the rate at which the volume of the balloon is increasing when its radius is 5 cm.

7. If $V = 0{\cdot}3z^3 - 0{\cdot}2z$, calculate $\frac{dV}{dt}$ in terms of z and $\frac{dz}{dt}$. Find $\frac{dV}{dt}$ if $z = 1{\cdot}3$ and $\frac{dz}{dt} = 0{\cdot}2$.

8. A circular metal disc of radius 4 cm. is being heated so that the radius is increasing at the rate of 0·01 cm. per hr.; find the rate at which the surface area is expanding.

9. If $y = x^3$, prove that $\frac{dx}{dt} = \frac{1}{3x^2}\frac{dy}{dt}$.

Hence calculate $\frac{dx}{dt}$, given that $\frac{dy}{dt} = 3$ cub. metres per hour and $x = 7$ metres.

10. Given that $y = x^2 + 1$, prove that $\frac{dx}{dt} = \frac{1}{2x}\frac{dy}{dt}$.

Hence calculate $\frac{dx}{dt}$, given that $\frac{dy}{dt} = 7$ sq. ml. per hour and $x = 120$ yards.

11. If $y = x^2 + x + 1$, $\frac{dy}{dt} = 8$ sq. feet per second and $x = 4$ inches, calculate $\frac{dx}{dt}$ in yards per minute.

12. $y = 5x^4 - 2x^3 - 7$, $\frac{dy}{dt} = 7$, $x = 8$; find $\frac{dx}{dt}$.

13. $x^2 + y^2 = 100$; prove that $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$.

Hence express $\frac{dy}{dt}$ in terms of x, y, $\frac{dx}{dt}$, and calculate $\frac{dy}{dt}$ in cm. per sec., given that $x = 6$ cm., $y = 8$ cm. and $\frac{dx}{dt} = 5$ cm. per sec.

14. Two men P and Q walk along two roads OA and OB at right angles to one another, the former at 4 m.p.h. and the latter at 3 m.p.h. They start simultaneously from O. Calculate the distance $r = PQ$ between the men after t hours.

Show that the rate $\left(\frac{dr}{dt}\right)$ at which the distance between the men is increasing is constant, and find this rate.

15. (i) If in Qu. 14, P passes O at 12 o'clock and Q passes O at 1 o'clock, their rates being as before, calculate the distance r between the men at 3 o'clock.

(ii) Calculate the distance r between them at t o'clock.

(iii) Find $\frac{dr}{dt}$ in terms of r and t, and hence find the rate at which the distance between P and Q is increasing at t o'clock.

(iv) Find their rate of separation at 3 o'clock.

16. $x^2+y^2=16$, $\frac{dx}{dt}=2$ in. per sec., $x=3$ in.; find $\frac{dy}{dt}$.

17. $y^2=36x$, $\frac{dy}{dt}=90$ ft. per sec., $y=40$ ft.; find $\frac{dx}{dt}$.

18. $27y^2=4(x-3)^3$, $\frac{dx}{dt}=11$, $x=6$; find $\frac{dy}{dt}$.

19. $x^3+y^3=35$, $\frac{dy}{dt}=50$ metres per min., $y=2$ metres; find $\frac{dx}{dt}$.

20. $y^3-7x^9=118$, $\frac{dy}{dt}=8$, $x=1$; find $\frac{dx}{dt}$.

21. Find the equation to the tangent at the point (1, 4) of the curve $y=3x^2+1$.

22. Find the equation to the tangent at the origin of the curve $y=5x^2+4x$.

23. Find the length of the subtangent at the point (1, 13) of the curve $y=4x^2+2x+7$.

24. Find the length of the subtangent at the point (2, -11) of the curve $y=5-2x-3x^2$.

25. Find the length of the subtangent at the point where the curve $y=ax^2+bx+c$ crosses the axis of y.

26. Find the length of the subnormal at the point (2, 17) of the curve $y=3x^2+5$.

27. Find the length of the subnormal at the point where the curve $y=5x^3-2x+1$ cuts the axis of y.

28. Find the ratio of the lengths of the subtangent and the subnormal at the point where the curve $y=x^2+x+1$ cuts the axis of y.

29. Find the radius of curvature at the origin of the curve $y=3x^2+5x$.

30. Find the radius of curvature at the vertex of the parabola $16y=x^2$.

31. A pulley is fixed 15 ft. above the ground, over which passes a rope 30 ft. long, with one end attached to a weight which can hang freely, and the other end is held by a man at a height of 3 feet from the ground. The man walks horizontally away from beneath the pulley at the rate of 3 ft. per sec. Find the rate at which the weight rises when it is 10 ft. above the ground.

32. Two pulleys are in the same horizontal line and a distance of 4 feet apart. A rope passes over both pulleys, and to its ends equal masses are attached, and another mass is attached to its middle point. Compare the rate of falling of the central mass with the rate of rising of either of the others when the central mass has descended 1 foot.

33 A trough with its ends ABC and A′B′C′ vertical and its upper edges horizontal has AB = BC, AC horizontal, and ABC = 90°. If AB = BC = 1 foot, calculate the volume of water which the trough will hold, if its length be 5 feet.

(i) Calculate the volume V of water in the trough if the depth be x inches.

(ii) Hence calculate $\frac{dV}{dt}$ in terms of x and $\frac{dx}{dt}$.

(iii) A tap runs water into the trough at the rate of $\frac{1}{2}$ a cubic foot per minute. Calculate the rate at which the depth of water is increasing in the trough when the water is 3 inches deep.

34. In Qu. 33, a distance y in. along BC has been wetted. Calculate V in terms of y. Hence find $\frac{dV}{dt}$ in terms of $\frac{dy}{dt}$ and y. Hence find the rate at which the edge BC is being wetted, when the water is 3 in. deep.

35. In Qu. 33, the area of the end ABC which is wet is S sq. in. Calculate V in terms of S. Hence find the rate in square inches per minute at which the end ABC is being wetted.

36. An inverted right circular cone is held vertex downwards. The semi-vertical angle is 20°.

(i) State the volume V cu. in. when there is a depth x in. of water.

(ii) Hence calculate $\frac{dV}{dt}$ in terms of x and $\frac{dx}{dt}$.

(iii) Water is poured into this cone at the rate of 1 cu. ft. per sec. Find the rate at which the depth is increasing when the water in the cone is 5 inches deep.

37. Calculate in Qu. 36 the wetted surface S in terms of x Hence express $\frac{dS}{dt}$ in terms of x and $\frac{dx}{dt}$.

Thereafter calculate the ratio of $\dfrac{\frac{dS}{dt}}{\frac{dV}{dt}}$ in terms of x.

From this last calculation deduce the rate at which the surface is being wetted when the water in the cone is 5 in. deep.

38. If $y=(1+x)^{17}$, calculate $\frac{dy}{dx}$.

39. If $y=(3x-4)^{10}$, calculate $\frac{dy}{dx}$.

40. If $y=(5-7x)^9$, calculate $\frac{dy}{dx}$.

41. If $y=(1+x)^{10}$, calculate $\frac{dy}{dt}$ in terms of x and $\frac{dx}{dt}$.

$$\left[N.B.\quad \frac{dy}{dt}=\frac{d(1+x)^{10}}{d(1+x)}\times\frac{d(1+x)}{dx}\times\frac{dx}{dt}.\right]$$

42. If $y=(1+x^2)^{10}$, find $\frac{dy}{dx}$. **43.** If $y=(3-4x^2)^7$, find $\frac{dy}{dx}$.

44. If $y=(x^3-x^2+3)^{15}$, find $\frac{dy}{dx}$.

45. If $y=(3x-4x^4)^9$, find $\frac{dy}{dt}$ in terms of x and $\frac{dx}{dt}$.

46. The volume V cubic feet of water in a hemispherical pot of radius a feet and depth x feet is given by $V=\pi\left(ax^2-\frac{x^3}{3}\right)$.

Water is discharged from a tap into a hemispherical pot of radius 3 feet at the rate of 1 cubic foot per second. If x feet be the depth of water in the pot, find the rate $\frac{dx}{dt}$ at which the depth of water is increasing. Graph $\frac{dx}{dt}$ against x on squared paper, and hence show that the depth increases very rapidly at first and more slowly as the depth increases.

Calculate the rate at which the depth of water is increasing when the water is 2 feet deep in the pot.

47. An instrument to time the boiling of eggs consists of two hollow glass cones of semivertical angle 30° with their apices together and axes in a straight line, and sand runs at a uniform speed from one to the other through the common vertex at the rate of 0·5 c.c. per min. If the height of each cone is 3 cm. and at the beginning of its flow the sand forms a cone of depth 2 cm., find the rate of decrease of the depth of the sand in the upper cone when its depth is 1 cm. and the corresponding rate of increase of the depth of sand in the lower cone.

CHAPTER V.

DIFFERENTIATION OF NEGATIVE AND FRACTIONAL POWERS.

1. Differentiation of negative integral powers of x.

We have seen in Chap. II. that $\frac{dx^n}{dx}=nx^{n-1}$, where n is a positive integer, and we should *expect* a generalization of this law to hold when n is a negative integer. Thus we should *expect* $\frac{dx^{-2}}{dx}=-2x^{-3}$; $\frac{dx^{-3}}{dx}=-3x^{-4}$; $\frac{dx^{-4}}{dx}=-4x^{-5}$; and so on. We shall find that this is the case, and we shall begin by working out a few easy instances.

Example 1. *To find the differential of* x^{-2}.

Let $$y=\frac{1}{x^2}.$$

$$\therefore\ y+\delta y=\frac{1}{(x+\delta x)^2}.$$

$$\therefore\ \delta y=\frac{1}{(x+\delta x)^2}-\frac{1}{x^2}$$

$$=\frac{x^2-(x+\delta x)^2}{x^2(x+\delta x)^2}$$

$$=\frac{x^2-x^2-2x\,.\,\delta x-(\delta x)^2}{x^2(x+\delta x)^2}$$

$$=-\frac{2x\,.\,\delta x+(\delta x)^2}{x^2(x+\delta x)^2}.$$

$$\therefore\ \frac{\delta y}{\delta x}=-\frac{2x+\delta x}{x^2(x+\delta x)^2}.$$

If we now proceed to the limit defined by δx becoming indefinitely small, we obtain

$$\frac{dy}{dx} = -\frac{2x}{x^2 \,.\, x^2}$$

$$= -\frac{2}{x^3}$$

$$= -2x^{-3}.$$

N. B.—This is in conformity with our *à priori* expectation, viz.

$$\frac{dx^{\ n}}{dx} = -nx^{-n-1}.$$

Example 2. (i) *To find in terms of* x_1 *and* x_2 *the gradient of the line joining the points* (x, y) *and* (x_1, y_1) *on the curve* $y = x^{-4}$.

We have

$$y = \frac{1}{x^4},$$

$$y_1 = \frac{1}{x_1{}^4}.$$

$$\therefore\ y_1 - y = \frac{1}{x_1{}^4} - \frac{1}{x^4}$$

$$= \frac{x^4 - x_1{}^4}{x^4 x_1{}^4}$$

$$= \frac{(x - x_1)(x^3 + x^2 x_1 + x x_1{}^2 + x_1{}^3)}{x^4 x_1{}^4}$$

$$\therefore\ \frac{y_1 - y}{x_1 - x} = -\frac{x^3 + x^2 x_1 + x x_1{}^2 + x_1{}^3}{x^4 x_1{}^4}.$$

Hence the gradient of the line joining the points (x, y) and (x_1, y_1) on the curve $y = x^{-4}$ is

$$-\frac{x^3 + x^2 x_1 + x x_1{}^2 + x_1{}^3}{x^4 x_1{}^4}.$$

(ii) *To deduce the gradient of the tangent at the point* (x, y) *on the curve* $y = x^{-4}$.

Putting $x_1 = x$ in the above result, we obtain that the gradient of the tangent at the point (x, y) is

$$-\frac{4x^3}{x^8}, \quad i.e. \ -4x^{-5}.$$

Hence,

$$\frac{dy}{dx} = -4x^{-5}, \quad \text{where } y = x^{-4}.$$

2. General proof that $\dfrac{dx^{-n}}{dx} = -nx^{-n-1}$, where n is a positive integer.

We shall now give the general proof that our *à priori* expectation of $\dfrac{dx^{-n}}{dx}$ being equal to $-nx^{-n-1}$ is fulfilled.

(i) *To find in terms of x and x_1 the gradient of the line joining the points (x, y) and (x_1, y_1) on the curve $y = x^{-n}$, where n is a positive integer.*

We have
$$y = \frac{1}{x^n},$$
$$y_1 = \frac{1}{x_1{}^n}.$$
$$\therefore\ y_1 - y = \frac{1}{x_1{}^n} - \frac{1}{x^n}$$
$$= \frac{x^n - x_1{}^n}{x^n x_1{}^n}$$
$$= \frac{(x - x_1)(x^{n-1} + x^{n-2}x_1 + x^{n-3}x_1{}^2 + \dots + x_1{}^{n-1})}{x^n x_1{}^n}.$$
$$\therefore\ \frac{y_1 - y}{x_1 - x} = -\frac{x^{n-1} + x^{n-2}x_1 + x^{n-3}x_1{}^2 + \dots + x_1{}^{n-1}}{x^n x_1{}^n}.$$

Hence the gradient of the line joining the points (x, y) and (x_1, y_1) on the curve $y = x^{-n}$ is
$$-\frac{x^{n-1} + x^{n-2}x_1 + x^{n-3}x_1{}^2 + \dots + x_1{}^{n-1}}{x^n x_1{}^n}.$$

(ii) *To deduce the gradient of the tangent at the point (x, y) on the curve $y = x^{-n}$.*

Putting $x_1 = x$ in the above result, we obtain that the gradient of the tangent at the point (x, y) is $-\dfrac{nx^{n-1}}{x^{2n}}$, *i.e.* $-nx^{-(n+1)}$.

Hence, $\dfrac{dy}{dx} = -nx^{-(n+1)}$, where $y = x^{-n}$

Example 1. Differentiate $y = \dfrac{3}{x^2} + \dfrac{7}{x^3}$.

We have
$$y = 3x^{-2} + 7x^{-3};$$
$$\therefore\ \frac{dy}{dx} = -6x^{-3} - 21x^{-4} = -\frac{6}{x^3} - \frac{21}{x^4}.$$

Example 2. *Find the least value which can be obtained by adding a number to its recripocal.*

We obtain the following table of values where $y = x + \frac{1}{x}$:

x	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1
y	$30\frac{1}{30}$	$20\frac{1}{20}$	$10\frac{1}{10}$	$5\frac{1}{5}$	$4\frac{1}{4}$	$3\frac{1}{3}$	$2\frac{1}{2}$	2
x	2	3	4	5	10	20	30	40
y	$2\frac{1}{2}$	$3\frac{1}{3}$	$4\frac{1}{4}$	$5\frac{1}{5}$	$10\frac{1}{10}$	$20\frac{1}{20}$	$30\frac{1}{30}$	$40\frac{1}{40}$

We thus see that the sum of a positive number and its reciprocal begins by being very large, then diminishes to the neighbourhood of 2, and finally increases again. In fact the graph has the following form:

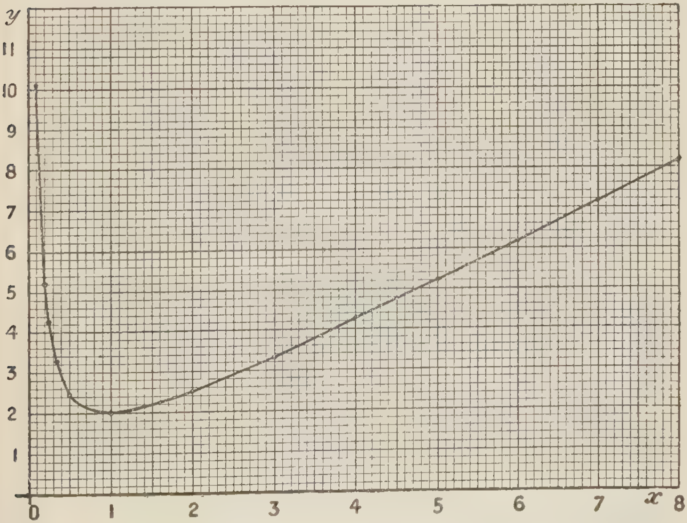


FIG. 36.

To find accurately where the function $y = x + \frac{1}{x}$ has its minimum value, we have to find where $\frac{dy}{dx}$ vanishes (*i.e.* where the tangent to the curve is parallel to $x'x$).

Now, since $$y = x + x^{-1},$$

$$\therefore\ \frac{dy}{dx} = 1 - x^{-2}.$$

$$\therefore\ \frac{dy}{dx} \text{ vanishes when } x = +1 \text{ or } -1.$$

As we are discussing only positive values of x we ignore $x = -1$, and retain only $x = 1$. Hence the sum of a number and its reciprocal is least when the number is 1 and the sum is 2.

Example 3. *A man wants to enclose an acre of land with a rectangular fence. How can he do it most economically?*

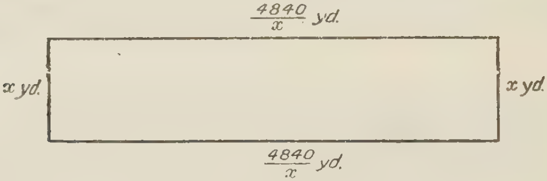


FIG. 37.

Let one side of the field be x yards.

$$\therefore \text{ the other side is } \frac{4840}{x} \text{ yds.}$$

Hence the total length l of fence required is given by

$$l = 2\left(x + \frac{4840}{x}\right).$$

Plotting l against x we get a graph of the form (Fig 38).

The graph shows that the "turning-value" is a minimum.

Now $$\frac{dl}{dx} = 2\left(1 - \frac{4840}{x^2}\right).$$

Hence $\frac{dl}{dx}$ vanishes when $x^2 = 4840$,

$$i.e.\ x = \sqrt{4840},$$

i.e. the field must be square, and the side must be of length $69\frac{1}{2}$ yards (very approximately).

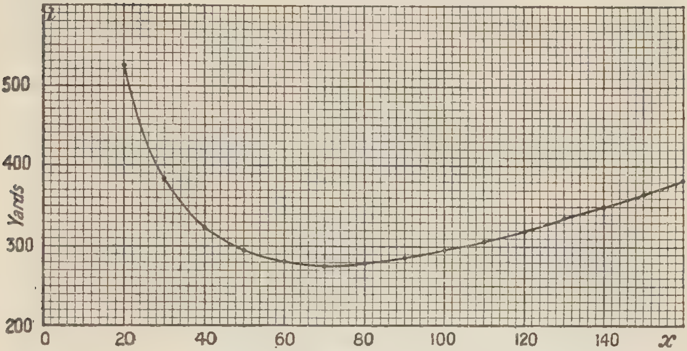


FIG. 38.

Example 4. *At any point* P *of the curve* $y=x^{-4}$, *the tangent at* P *meets the axis of* x *in* T *and* N *is the foot of the perpendicular let fall from* P *on to* $x'x$. *Prove that* 4NT = ON.

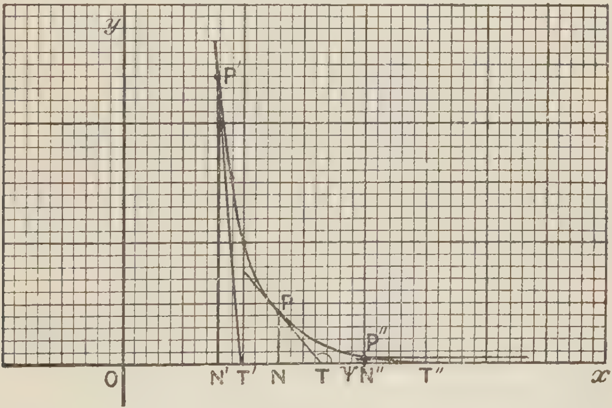


FIG. 39.

We have
$$y=x^{-4}.$$
$$\therefore\ \tan\psi=\frac{dy}{dx}=-4x^{-5}.$$
$$\therefore\ \mathrm{NT}=-\mathrm{PN}\cot\psi=-y\times\left(-\frac{x^5}{4}\right)=\frac{x}{4}=\frac{\mathrm{ON}}{4}.$$

Example 5. *Prove that* $y=x^{-3}$ *satisfies the differential equation* $x\dfrac{dy}{dx}+3y=0.$

For
$$\frac{dy}{dx}=-3x^{-4}.$$

$$\therefore\ x\frac{dy}{dx}+3y=x\times(-3x^{-4})+3\times x^{-3}$$
$$=0.$$

3. Differentiation of $y=x^{\frac{m}{n}}$, where m and n are positive integers.

(i) *To find in terms of* x *and* y *the gradient of the line joining the points* (x, y) *and* (x_1, y_1) on *the curve* $y=x^{\frac{m}{n}}$, *where* m *and* n *are positive integers.*

We have
$$y=x^{\frac{m}{n}}.$$
$$\therefore\ y^n=x^m \quad\text{and}\quad y_1{}^n=x_1{}^m.$$
$$\therefore\ y_1{}^n-y^n=x_1{}^m-x^m.$$
$$\therefore\ (y_1-y)(y_1{}^{n-1}+y_1{}^{n-2}y+y_1{}^{n-3}y^2+\ldots+y^{n-1})$$
$$=(x_1-x)(x_1{}^{m-1}+x_1{}^{m-2}x+x_1{}^{m-3}x^2+\ldots+x^{m-1}).$$
$$\therefore\ \frac{y_1-y}{x_1-x}=\frac{x_1{}^{m-1}+x_1{}^{m-2}x+x_1{}^{m-3}x^2+\ldots+x^{m-1}}{y_1{}^{n-1}+y_1{}^{n-2}y+y_1{}^{n-3}y^2+\ldots+y^{n-1}}.$$

Hence the gradient of the line joining the points (x, y) and (x_1, y_1) on the curve $y=x^{\frac{m}{n}}$ is
$$\frac{x_1{}^{m-1}+x_1{}^{m-2}x+x_1{}^{m-3}x^2+\ldots+x^{m-1}}{y_1{}^{n-1}+y_1{}^{n-2}y+y_1{}^{n-3}y^2+\ldots+y^{n-1}}.$$

(ii) *To deduce the gradient of the tangent at the point* (x, y) *on the curve* $y=x^{\frac{m}{n}}$, *where* m *and* n *are positive integers.*

Putting $x_1=x$ and $y_1=y$ in the above result, we obtain that the gradient of the tangent at the point (x, y) is
$$\frac{mx^{m-1}}{ny^{n-1}},\quad i.e.\ \frac{m}{n}\cdot\frac{x^{m-1}}{\left(x^{\frac{m}{n}}\right)^{n-1}},\quad i.e.\ \frac{m}{n}x^{\frac{m}{n}-1}.$$

Otherwise $\dfrac{dy}{dx}=\dfrac{m}{n}x^{\frac{m}{n}-1}$, where $y=x^{\frac{m}{n}}$.

4. Differentiation of $x^{\frac{m}{n}}$, where m and n are positive or negative integers.

Let
$$y=x^{\frac{m}{n}}. \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$\therefore\ y^n=x^m. \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$
$$\therefore\ \frac{dy^n}{dx}=\frac{dx^m}{dx}.$$
$$\therefore\ \frac{dy}{dx}\times\frac{dy^n}{dy}=\frac{dx^m}{dx}\quad\text{(by Chap. IV. Art. 1).}$$
$$\therefore\ \frac{dy}{dx}\times ny^{n-1}=mx^{m-1}$$

($\because$ m and n are positive or negative integers by Chap. II. Art. 6 and Chap. V. Art. 2).

$$\therefore\ \frac{dy}{dx}=\frac{m}{n}\frac{x^{m-1}}{y^{n-1}}.$$
$$\therefore\ \frac{dy}{dx}=\frac{m}{n}x^{m-1}\div\left(x^{\frac{m}{n}}\right)^{n-1},\ \text{by (1)},$$
$$=\frac{m}{n}x^{m-1}\div x^{\frac{m(n-1)}{n}}$$
$$=\frac{m}{n}\frac{x^{m-1}}{x^{m-\frac{m}{n}}}$$
$$=\frac{m}{n}x^{\frac{m}{n}-1}$$

N.B.—This is the same formula as when the index is a positive or negative integer.

Example 1. *Prove that $y=3+4x^{\frac{1}{2}}$ is a solution of the differential equation $y\dfrac{dy}{dx}=3\dfrac{dy}{dx}+8$.*

$$\because\ y=3+4x^{\frac{1}{2}},\quad\therefore\ \frac{dy}{dx}=2x^{-\frac{1}{2}}.$$

Hence $\quad \text{L.H.S.} = y\dfrac{dy}{dx} = (3 + 4x^{\frac{1}{2}}) \times 2x^{-\frac{1}{2}} = 6x^{-\frac{1}{2}} + 8.$

Also $\quad \text{R.H.S.} = 3\dfrac{dy}{dx} + 8 = 6x^{-\frac{1}{2}} + 8.$

Hence $\quad$ R.H.S. = L.H.S.

Example 2. *A spherical elastic balloon is being filled with water at the rate of 5 c.c. per second. Find the rate at which the radius is increasing after 3 seconds.*

Let V c.c. be the volume and r cm. be the radius.

$$\therefore\ V = \tfrac{4}{3}\pi r^3.$$

$$\therefore\ r = \left(\frac{3}{4\pi}\right)^{\frac{1}{3}} V^{\frac{1}{3}}.$$

$$\therefore\ r = 0{\cdot}6203 V^{\frac{1}{3}}.$$

$$\therefore\ \frac{dr}{dt} = 0{\cdot}2068 V^{-\frac{2}{3}} \frac{dV}{dt}. \quad \text{..............................(1)}$$

Now $\quad \dfrac{dV}{dt} = 5$ c.c. per sec. ....(2)

Also $\quad V = 15$ c.c. after 3 sec. ....(3)

$$\therefore\ \frac{dr}{dt} = 0{\cdot}2068 \times \frac{5}{15^{\frac{2}{3}}} \text{ cm. per sec. (after 3 sec.)}$$

$$= 0{\cdot}17 \text{ cm. per sec. (after 3 sec.).}$$

5. Differentiation by substitution.

We are frequently required to differentiate more complicated expressions than the above, and we use the "Method of Substitution."

A few examples will make the method of procedure plain.

Example 1. *Differentiate* $\sqrt{1 + x}$.

We have $\quad y = (1 + x)^{\frac{1}{2}}.$

Let $\quad z = 1 + x.$

Then $\quad \dfrac{dy}{dx} = \dfrac{dy}{dz} \times \dfrac{dz}{dx}$ (by Chap. IV. Art. 1). ....(1)

Now $\quad \dfrac{dy}{dz} = \dfrac{dz^{\frac{1}{2}}}{dz} = \dfrac{1}{2\sqrt{z}},$ ....(2)

and $\quad \dfrac{dz}{dx} = \dfrac{d(1 + x)}{dx} = 1.$ ....(3)

Hence, by (1), (2) and (3),

$$\frac{dy}{dx}=\frac{1}{2\sqrt{z}}$$

$$=\frac{1}{2\sqrt{1+x}}.$$

Example 2. *Differentiate* $\sqrt{1-x^2}$.

Let $y=(1-x^2)^{\frac{1}{2}}$.

Let $z=1-x^2$.

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}\text{ (by Chap. IV. Art. 1). } \qquad\ldots\ldots\ldots\ldots(1)$$

Now
$$\frac{dy}{dz}=\frac{dz^{\frac{1}{2}}}{dz}=\frac{1}{2\sqrt{z}}=\frac{1}{2\sqrt{1-x^2}}, \qquad\ldots\ldots\ldots\ldots(2)$$

$$\frac{dz}{dx}=\frac{d(1-x^2)}{dx}=-2x. \qquad\ldots\ldots\ldots\ldots(3)$$

$\therefore$ by (1), (2) and (3), $\dfrac{dy}{dx}=-\dfrac{x}{\sqrt{1-x^2}}$.

6. Usually, in practice, after we have obtained a certain degree of expertness, we do not actually make the substitution described in the last article. A few illustrations will make the practical method of procedure plain.

Example 1. *To differentiate* $\sqrt{1+x^2}$.

We have
$$\frac{d(1+x^2)^{\frac{1}{2}}}{dx}=\frac{d(1+x^2)^{\frac{1}{2}}}{d(1+x^2)}\times\frac{d(1+x^2)}{dx}$$

$$=\tfrac{1}{2}(1+x^2)^{-\frac{1}{2}}\times 2x$$

$$=\frac{x}{\sqrt{1+x^2}}.$$

Example 2. *To differentiate* $(3x^2+4x^3)^{\frac{3}{4}}$.

We have
$$\frac{d(3x^2+4x^3)^{\frac{3}{4}}}{dx}=\frac{d(3x^2+4x^3)^{\frac{3}{4}}}{d(3x^2+4x^3)}\times\frac{d(3x^2+4x^3)}{dx}$$

$$=\tfrac{3}{4}(3x^2+4x^3)^{-\frac{1}{4}}\times(6x+12x^2)$$

$$=\frac{9(x+2x^2)}{2(3x^2+4x^3)^{\frac{1}{4}}}.$$

Example 3. *The hypotenuse of a right-angled triangle is 10 cm. in length. Find what must be the lengths of the sides, so that the perimeter of the triangle may be as large as possible.*

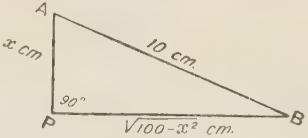


FIG. 40.

Take the sides of the triangle as shown, and let P cm. denote the perimeter.

$$\therefore\ P = 10 + x + \sqrt{100 - x^2}.$$

Hence, plotting P against x from $x = 0$ to $x = 10$, we get the graph as shown.

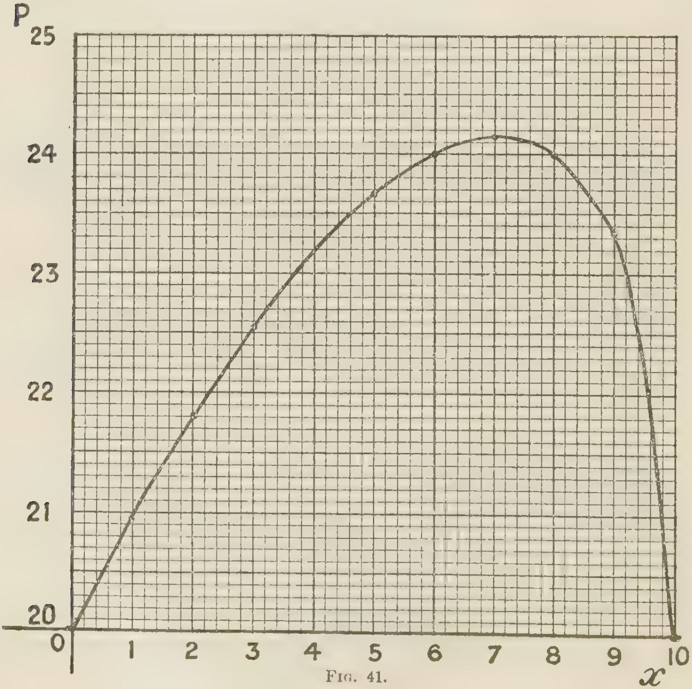


FIG. 41.

The graph shows that the turning-value is a maximum. We have :

$$\frac{dP}{dx}=1+\frac{d}{dx}(100-x^2)^{\frac{1}{2}}$$

$$=1+\frac{d(100-x^2)^{\frac{1}{2}}}{d(100-x^2)}\times\frac{d(100-x^2)}{dx}$$

(or putting $z=100-x^2$ if the student so desires)

$$=1+\tfrac{1}{2}(100-x^2)^{-\frac{1}{2}}\times(-2x)$$

$$=1-\frac{x}{\sqrt{100-x^2}}.$$

Hence $\frac{dP}{dx}$ vanishes when $x^2=50$, *i.e.* when $x=5\sqrt{2}=7\cdot07$. In this case the triangle is isosceles.

The graph verifies our conclusion.

EXAMPLES V.

1. Differentiate x^{-3}, $\frac{1}{x^5}$, $\frac{7}{x^9}$, $9x^{-4}$, $\frac{4}{3}x^{-3}$, x^{-5}, $3+4x^{-1}+5x^{-2}$, $3x+4+5x^{-1}$, x^{-n+3}.

2. Differentiate $x^{\frac{3}{2}}$, $\frac{2}{3}x^{\frac{9}{4}}$, $3+5x^{\frac{3}{5}}$, $x^{-\frac{4}{5}}$, $\frac{3}{5}x^{-\frac{5}{3}}$, $7x^{-\frac{1}{2}}$, $ax^{\frac{1}{2}}+bx^{-\frac{1}{2}}$, $4x^{\frac{4}{3}}+5x^{\frac{1}{3}}+7x^{-\frac{2}{3}}$.

3. If $y=3x^{\frac{1}{3}}$, find $\frac{dy}{dx}$ when $x=27$.

4. If $y=3x^{\frac{1}{2}}+4x^{-\frac{1}{2}}$, find $\frac{dy}{dx}$ when $x=7$ correct to two places of decimals.

5. Prove that $y=x+\frac{1}{x}$ satisfies the differential equation

$$x\frac{dy}{dx}+y=2x.$$

6. Prove that $y=x^{\frac{1}{2}}+x^{-\frac{1}{2}}$ satisfies the differential equation

$$2x\frac{dy}{dx}+y=2\sqrt{x}.$$

7. In any curve the length of the subtangent is given by the formula $t=y\div\frac{dy}{dx}$.

(i) Find t in terms of x in the case of the curve $xy=4$.
(ii) Find t in terms of y in the case of the curve $xy=4$.
(iii) Find t when $x=1$ and $y=4$ in the case of the curve $xy=4$.

8. In any curve the length of the subnormal n is given by the formula $n=y\times\frac{dy}{dx}$.

(i) Find n in terms of x in the case of the curve $xy=4$.

(ii) Find n in terms of y in the case of the curve $xy=4$.

(iii) Find n when $x=1$ and $y=4$ in the case of the curve $xy=4$.

9. Find t and n in terms of x in the case of the curve $yx^{\frac{3}{2}}=1$.

10. In the case of the curve $y^5=x^3$, prove that $\frac{t}{n}=\frac{25}{9}x^{\frac{1}{5}}y$.

11. Differentiate by the *method of substitution* $(1+x^2)^{\frac{1}{2}}$, $(1+x^2)^{\frac{3}{2}}$, $(1+x^2)^{-\frac{1}{2}}$, $(3+4x)^{\frac{1}{2}}$, $(3+4x)^{\frac{1}{4}}$, $(3+4x)^{-\frac{1}{3}}$, $(2x^2-3x+7)^{\frac{1}{4}}$, $(2x^2-3x+7)^{\frac{3}{2}}$, $(1-x^2)^{\frac{1}{2}}$, $(1-x^2)^{\frac{3}{2}}$, $(1-x^2)^{-1}$, $(4-5x-6x^2)^{-2}$, $(3-2x-5x^2)^{-\frac{1}{2}}$.

12. Prove that $y=\sqrt{1+x^2}$ satisfies the differential equation $y\frac{dy}{dx}=x$.

13. If $y=\sqrt{5+x^2}$, find $\frac{dy}{dx}$ when $x=2$.

14. Prove that $y=(1-x^2)^{\frac{3}{2}}$ satisfies the differential equation $(1-x^2)\frac{dy}{dx}+3xy=0$.

15. If $y=(37+4x-10x^2)^{\frac{3}{2}}$, find $\frac{dy}{dx}$ correct to two places of decimals when $x=2$.

16. Sketch the curve $y=\sqrt{1+x^2}$.

If P be a point on the curve, PN perpendicular to $x'x$, and PG the normal at P (N and G lying on $x'x$), prove that PN is the perpendicular bisector of OG.

17. A man intends to buy sheet zinc, of uniform width and length 1760 yds., to make a closed water-duct a mile long, whose rectangular cross-section is 4 square feet. He wishes to find out the most economical width.

If y feet be the width of the sheet zinc, and x feet the depth of the duct, express y in terms of x. Plot y against x to see the nature of the relative variations of y and x.

Use the Calculus to find the most economical dimensions for the cross-section.

18. A man wishes to make a water-duct, open at the top, 3 mls. long, out of a strip of sheet zinc of uniform width. The duct is to have a rectangular cross-section of area 6 sq. ft.

If y feet be the width of the sheet-zinc, and x feet the depth of the duct. express y in terms of x. Plot y against x to see the nature of the relative variations of y and x.

Use the Calculus to find the most economical dimensions for the cross-section.

19. A man wishes to make a rectangular cistern with a square base and closed at the top, so as to hold 1000 cubic feet of water. He wishes to line the cistern with sheet lead. Find the most economical dimensions for the cistern.

20. A man wishes to make a rectangular cistern with a square base and open at the top, so as to hold 1000 cubic feet of water. He wishes to line the cistern with sheet lead. Find the most economical dimensions for the cistern and the cost of lining it with lead at $4\frac{1}{2}$d. per square foot.

21. An iron boiler consists of a right circular cylinder closed at both ends. It is constructed so as to contain 80 cubic feet of water. Find the most economical dimensions so that the cost of iron shall be as small as possible.

22. A triangle is right-angled and its hypotenuse is always of length 10 inches. What is its maximum area ?

23. The sum of the two sides of a right-angled triangle is 12 cm. What is the minimum length of the hypotenuse ?

24. A photographic lens of 10 inches focal length forms an image of an object. What is the minimum possible distance between the object and image ?

[*Note.*—If the object, *e.g.* candle flame, be placed x inches on one side of the lens, and the image be caught y inches on the other side of the lens, it is proved in the Theory of Light that x and y are connected by the formula $\frac{1}{x}+\frac{1}{y}=\frac{1}{f}$, where f inches is the focal length of the lens.]

25. When is $x^{\frac{2}{\gamma}} - x^{1+\frac{1}{\gamma}}$ a maximum, γ being $1{\cdot}4$? Plot the values near the maximum value. (Bd. of Educ.)

26. If $y=2x+\frac{1{\cdot}5}{x}$, state what value of x will make y less than any other. (Bd. of Educ.)

27. In a certain vessel it happens to be true, within certain limits, that $V=1200\,h^{1\cdot 5}$, where h is the vertical draught in feet and V is the displacement in cubic feet. The vessel is being loaded up, and hence draws more water. Find the relative rate at which the displacement and draught are increasing when the vessel draws 20 ft. of water.

28. If $v=\frac{c}{r}$, prove that $\frac{v}{r}+\frac{dv}{dr}=0$.

29. The velocity of sound in air is $66{\cdot}3\sqrt{t}$ feet per second, where t is the absolute temperature Centigrade (that is, the ordinary temperature plus 273). Calculate the rate of rise in the velocity, per degree rise in the temperature, at 10° C.

30. What functions have got $x^{\frac{1}{2}}$, $3x^{\frac{3}{4}}$, $5x^{-2}$, $7x^{-3}$, $9x^{-2}+7x^{-3}$, $5x^{\frac{1}{2}}+7x^{-\frac{1}{2}}$ respectively as differential coefficients ?

31. The electrical resistance of wires of the same length varies inversely as the square of the diameter.

The resistance of 1000 yd. of Eureka wire whose diameter is 0·024 inch is 1576 ohms. What is the rate of change of the resistance per 0·001 inch change in diameter at this diameter ?

32. A V-shaped cattle trough whose ends are equilateral triangles is to be made of iron of given thickness, and when full it is to contain 3 cubic feet of water. What must its dimensions be so that the weight of the trough shall be the least possible ?

33. The radius of a heated sphere is increasing at the rate of 0·0002 cm. per minute. Find the rate of increase of the surface when the radius of the sphere is 10 cm.

34. The radius of a heated sphere is increasing at the rate of 0·0002 cm. per minute. Find the rate of increase of the volume when the radius of the sphere is 10 cm.

35. Find the formula connecting the surface and the volume of a sphere.

If the surface is increasing at the rate of a sq. cm. per second, find the rate at which the volume is increasing.

36. A vessel in the shape of an inverted cone, whose semi-vertical angle is 27°, is being filled with water at the rate of 3 c.c. per sec. Find the volume of water in the cone when the depth is increasing at the rate of 2 cm. per sec.

37. A ship is sailing parallel with the shore and $\frac{1}{4}$ mile from it at the rate of 10 miles per hour. Find the rate at which it is increasing its distance from a man on the shore, when it is half a mile from him.

38. A man standing at one end of the Clifton Suspension Bridge watches a stone let fall from a point 200 feet away. Find the rate at which the stone is moving away from him after 2 seconds.

39. A man 6 feet high is standing at night 4 feet away from a mast, up which is being raised at the rate of 3 feet per sec. a lighted lantern. Find the rate at which the shadow of his head approaches the man when it is distant 4 feet from him.

40. Find the inclinations in degrees and minutes to the axis of x of the tangents to the curve $xy = 16$ at the points (8, 2) and (-2, -8). Illustrate with a rough graph.

41. The tangent at a point P on the curve $y = 5x^3$ meets the axis of x at T, and PN is the perpendicular from P on the axis of x. If O is the origin, show that ON is divided at T in a constant ratio. Illustrate with a rough graph.

42. The tangent at any point P to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ meets the axis of x in T, and PN is the perpendicular from P on the axis of x. Prove that ON . OT = 25, O being the origin. Illustrate with a rough graph.

43. Find the area of the rectangle of maximum area which can be inscribed in the ellipse whose equation is $\frac{x^2}{16}+\frac{y^2}{9}=1$, the sides of the rectangle being parallel to the axes.

44. A straight line is drawn parallel to the axis of y in the curve $y^2 = 25x$, and its distance from the axis is 48 mm. Find the area of the rectangle of maximum area which has two vertices on the curve and two on the straight line.

45. Find also the rectangle of maximum perimeter in similar circumstances to that of Question 44.

46. Brine of density 1·43 grams per c.c. flows at the rate of 2 cub. ft. per min. into a tank originally containing 10 cub. ft. of pure water. Find the rate of increase of density of the liquid in the tank after 5 minutes.

47. Two electric light bulbs of 32 and 16 candle-power respectively are placed 10 feet apart on a table. If the intensity of illumination varies inversely as the square of the distance, which point on the table between them is best lighted ?

48. A man is walking at the rate of 4 miles per hour towards a camera in which is a lens of 6 in. focus. When he is distant 50 feet from the camera, what must be the rate at which the ground glass is being "racked" out, to keep him in proper focus ?

49. In the curve $xy = c^2$ prove that the tangent at any point has that portion of it intercepted between the axes bisected by the point of contact.

50. What function is such that its derived function is always identical with the reciprocal of the function ?

51. A man rows up-stream a certain distance and back again to the starting point. Prove that the time taken will be a minimum if there is no current.

CHAPTER VI.

DIFFERENTIAL EQUATIONS AND INDEFINITE INTEGRALS.

1. Differential Equations.

In dealing with the differential calculus a perfectly new type of equation appears. It presents itself in the attempt to solve various problems that appear in practical life and scientific investigations.

Illustration 1. *Experiment shows that a body, falling in a vacuum, under the influence of gravity acquires a velocity proportional to the time of fall. To find the distance fallen in a given time.*

Put into algebraic form, our experimental result is $v = 32{\cdot}2t$, that is, in the language of the Calculus,

$$\frac{ds}{dt} = 32{\cdot}2t$$

(where s feet is the distance fallen, and t is measured in seconds).

This equation is called a "*differential equation,*" and we are required to express s in terms of t.

Illustration 2. *If we study the pressure to which the surface of a body is subjected as it descends in the sea, we find that we obtain a differential equation of the form*

$$\frac{dp}{dx} = w$$

(*where p is the pressure in known units of measurement, x is the depth, and w is a known value*).

Here again we have obtained a differential equation in p and x, and we have to express p in terms of x.

Illustration 3. *If we study the pressure to which the surface of a body is subjected as it ascends in the air, we find that we obtain a differential equation of the form*

$$\frac{dp}{dx} = -\mathrm{K}p$$

(*where p is the pressure in known units of measurement, x is the height, and* K *has a known constant value determined by experiment*).

Here we have obtained a differential equation in p and x of a totally different type from the last.

Illustration 4. *In finding the area* A *of a parabola we have to solve a differential equation of the form*

$$\frac{d\mathrm{A}}{dx} = 2a^{\frac{1}{2}}x^{\frac{1}{2}}.$$

Those acquainted with Electricity, Sound, Heat, etc., will find innumerable other differential equations arising, that have to be solved.

N.B.—Such expressions as x^2, $x+5$, x^2+2x-9, $(x^3+3x+2)^{\frac{1}{2}}$, etc., are called *functions of x*, and we find it convenient to use $f(x)$ to denote briefly and concisely the words "*function of x.*"

2. Examples of differential equations.

The differential equation

$$\frac{dy}{dx} = f(x)$$

asks the question : " *What function* (*or functions*) *of x are such that their differential coefficients are $f(x)$?* "

Consider the particular case

$$\frac{dy}{dx} = 3x^2.$$

What function (*or functions*) *of x are such that their differential coefficients are each $3x^2$?*

Plainly
$$\frac{dx^3}{dx} = 3x^2,$$

$$\frac{d}{dx}(x^3+5) = 3x^2,$$

$$\frac{d}{dx}(x^3-7)=3x^2,$$

$$\frac{d}{dx}(x^3+\tfrac{3}{4})=3x^2,$$

and in general $$\frac{d}{dx}(x^3+\mathrm{C})=3x^2,$$

where C is an arbitrary constant.

Hence $y=x^3+\mathrm{C}$ is a *very general* solution to the above differential equation.

Again consider the differential equation

$$\frac{dy}{dx}=x^4.$$

What function (or functions) of x are such that their differential coefficients are each x^4 ?

Plainly $$\frac{d}{dx}(\tfrac{1}{5}x^5)=x^4;$$

$$\frac{d}{dx}(\tfrac{1}{5}x^5+19)=x^4,$$

$$\frac{d}{dx}(\tfrac{1}{5}x^5-33)=x^4,$$

$$\frac{d}{dx}(\tfrac{1}{5}x^5-\tfrac{19}{8})=x^4,$$

and in general $$\frac{d}{dx}(\tfrac{1}{5}x^5+\mathrm{C})=x^4,$$

where C is an arbitrary constant.

Hence $y=\frac{1}{5}x^5+\mathrm{C}$ is a *very general* solution to the above differential equation.

3. To find the general solution to a simple differential equation of the type $\frac{dy}{dx}=f(x)$.

Let $y=\mathrm{F}(x)$ and $y=\mathrm{G}(x)$

be two possible solutions of the above differential equation.

$$\therefore \frac{d}{dx}\mathrm{F}(x)=f(x),$$

$$\frac{d}{dx}\mathrm{G}(x)=f(x).$$

Hence subtracting, we obtain

$$\frac{d}{dx}\mathrm{F}(x)-\frac{d}{dx}\mathrm{G}(x)=0.$$

$$\therefore \frac{d}{dx}\{\mathrm{F}(x)-\mathrm{G}(x)\}=0.$$

But the only function whose differential coefficient is always zero is a constant.

$$\therefore \mathrm{F}(x)-\mathrm{G}(x)=\mathrm{C}.$$

Hence two functions of x having always the same differential coefficient, must differ by a constant.

4. *Rule for finding the most general function whose differential coefficient is given.*

Find by inspection the most simple solution (i.e. the most simple function you can think of), whose differential coefficient is the given function, and add on an arbitrary constant.

An example will make the process clear. Knowledge of Differentiation enables us to find the simplest functions having given differential coefficients

Example. *Find the general solution of the differential equation*

$$\frac{dy}{dx}=x^2+x+1.$$

Inspection shows us that

$$x^2=\frac{d}{dx}\left(\frac{x^3}{3}\right),$$

$$x=\frac{d}{dx}\left(\frac{x^2}{2}\right),$$

$$1=\frac{d}{dx}(x).$$

$$\therefore x^2+x+1=\frac{d}{dx}\left(\frac{x^3}{3}+\frac{x^2}{2}+x\right).$$

Hence we may write the given differential equation as

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x^3}{3}+\frac{x^2}{2}+x\right).$$

Hence *one* value for y is $\frac{x^3}{3}+\frac{x^2}{2}+x$.

The general solution is therefore $y=\frac{x^3}{3}+\frac{x^2}{2}+x+C$.

5. Determination of the constant C.

We can determine the value of C in the case of a given problem from the data of the problem.

A few examples will make this clear.

Example 1. *Solve completely the differential equation*

$$\frac{dy}{dx}=x^2+x+1,$$

given that $y=3$ *when* $x=1$.

By the example of last article the general solution is

$$y=\frac{x^3}{3}+\frac{x^2}{2}+x+C. \qquad\qquad (1)$$

Substitute in (1), $x=1$ and $y=3$.

$$\therefore\ 3=\tfrac{1}{3}+\tfrac{1}{2}+1+C.$$

$$\therefore\ C=1\tfrac{1}{6}. \qquad\qquad (2)$$

Hence, by (1) and (2),

$$y=\frac{x^3}{3}+\frac{x^2}{2}+x+1\tfrac{1}{6}$$

is the required solution.

Example 2. *The velocity of a body let fall from rest in a vacuum is given by* $v=32t$ *(approx.), (*t *being measured in seconds and* v *in feet per second). Find the distance fallen in* t *seconds.*

Let s feet be the distance fallen.

Hence, as usual, $v=32t$ may be written

$$\frac{ds}{dt}=32t.$$

$$\therefore\ \frac{ds}{dt}=\frac{d}{dt}(16t^2), \text{ by inspection.}$$

$$\therefore\ s=C+16t^2. \qquad\qquad (1)$$

Now, when $t=0$, s is also $=0$. Substitute in (1).

$$\therefore\ C=0. \qquad\qquad (2)$$

$\therefore$ by (1) and (2), $s=16t^2$.

Example 3. *It is known that under certain influences a body moves with a velocity* $v = 16 - 7t$, *where* t *is measured in seconds and* v *in feet per sec. Find the distance traversed from a fixed point after* t *seconds, given that the body is 5 feet from the point at the end of the third second of the motion.*

As before, $v = 16 - 7t$ may be written

$$\frac{ds}{dt} = 16 - 7t.$$

$$\therefore \frac{ds}{dt} = \frac{d}{dt}(16t - \tfrac{7}{2}t^2), \text{ by inspection.}$$

$$\therefore s = C + 16t - \tfrac{7}{2}t^2. \quad \text{.......................................(1)}$$

Now, when $t = 3$, $s = 5$. Substitute in (1).

$$\therefore 5 = C + 48 - \tfrac{63}{2}.$$

$$\therefore C = -11\tfrac{1}{2}.$$

$$\therefore s = -11\tfrac{1}{2} + 16t - \tfrac{7}{2}t^2.$$

6. Notation.

Suppose that $f(x)$ and $\phi(x)$ are two functions of x connected by the relation

$$f(x) = \frac{d\phi(x)}{dx}.$$

Here $f(x)$ is expressed in terms of $\phi(x)$.

It is found convenient to have some method of expressing $\phi(x)$ explicitly in terms of $f(x)$, and we therefore write

$$\phi(x) = \int f(x)\,dx.$$

The symbol "$\int$" is called the symbol of "integration." It is really an elongated "S," the first letter of the word "sum," and arises from the fact that the process of "integration" first presented itself in trying to calculate an area by finding the limit of the sum of a number of very small areas.

$\int f(x)\,dx$ asks the question: "*What function of* x *has* $f(x)$ *as its differential coefficient?*" Hence "integration" must be regarded in the first instance as the reverse of "differentiation."

We therefore have the following table:

$$\because \frac{d}{dx}\left(\frac{x^2}{2}\right) = x, \qquad \therefore \int x\,dx = \frac{x^2}{2}.$$

$$\because \frac{d}{dx}\left(\frac{x^3}{3}\right)=x^2, \qquad \therefore \int x^2\,dx=\frac{x^3}{3}.$$

$$\because \frac{d}{dx}\left(\frac{x^4}{4}\right)=x^3, \qquad \therefore \int x^3\,dx=\frac{x^4}{4}.$$

$$\because \frac{d}{dx}\left(\frac{x^5}{5}\right)=x^4, \qquad \therefore \int x^4\,dx=\frac{x^5}{5}.$$

……

In general, $\because \dfrac{d}{dx}\dfrac{x^{n+1}}{n+1}=x^n, \qquad \therefore \displaystyle\int x^n\,dx=\frac{x^{n+1}}{n+1}.$

This last result should be committed to memory.

[*N.B.*—The case $\int x^{-1}dx$ is omitted here, and is discussed in Part II. of this work under Logarithms.]

Example 1. *Find* $\int (x+x^2+x^3)\,dx$.

$\int (x+x^2+x^3)\,dx$ means that we are looking for that function of x whose differential coefficient is $x+x^2+x^3$.

$$\therefore \int (x+x^2+x^3)\,dx = \int x\,.\,dx + \int x^2\,dx + \int x^3\,dx$$

$$=\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}.$$

Example 2. *Find* $\int (x^{\frac{1}{2}}+x^{-\frac{1}{2}})\,dx$.

$$\int (x^{\frac{1}{2}}+x^{-\frac{1}{2}})\,dx=\int x^{\frac{1}{2}}\,dx+\int x^{-\frac{1}{2}}\,dx$$

$$=\frac{x^{\frac{3}{2}}}{3/2}+\frac{x^{\frac{1}{2}}}{1/2} \text{ (by the general rule)}$$

$$=\tfrac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}.$$

7. Note on the value of $\int f(x)\,dx$.

$\int f(x)\,dx$ means that we are looking for a function whose differential coefficient is $f(x)$.

Suppose that $\phi(x)$ is such a function.

$$\therefore \frac{d\phi(x)}{dx}=f(x).$$

But also $\frac{d}{dx}\{\phi(x)+C\}=f(x)$, where C is an arbitrary constant.

Hence $\int f(x)\,dx=\phi(x)+C$, where $\frac{d\phi(x)}{dx}=f(x)$.

We thus see that $\int f(x)\,dx$ has not got a single unique value but an infinity of values, obtained by adding arbitrary constants to any one known value of $\int f(x)\,dx$. It is convenient, however, to use $\int f(x)\,dx$ to denote the simplest or most convenient value and to add on arbitrary constants where required. Thus the general value of $\int x^2 dx$ is $\frac{x^3}{3}+C$.

8. Integration of a differential equation.

Being given the differential equation

$$\frac{dy}{dx}=f(x),$$

we can now write its solution as

$$y=C+\int f(x)\,dx.$$

Example 1. *Solve the differential equation* $\frac{dy}{dx}=x^9$, *given that* $y=7$ *when* $x=0$.

We have
$$\frac{dy}{dx}=x^9.$$
$$\therefore\ y=C+\int x^9 dx.$$
$$\therefore\ y=C+\frac{x^{10}}{10}. \quad \ldots\ldots\ldots\ldots(1)$$

Also, since $y=7$ when $x=0$, we have from (1)
$$7=C. \quad \ldots\ldots\ldots\ldots(2)$$
$$\therefore\ y=7+\frac{x^{10}}{10}, \text{ by (1) and (2).}$$

Example 2. *Find the value of* y *when* $x=0$, *given that*
$$\frac{dy}{dx}=1{\cdot}2x^5-0{\cdot}5x$$
and that $y=2{\cdot}3$ *when* $x=-1$.

$$\because \frac{dy}{dx} = 1{\cdot}2x^5 - 0{\cdot}5x,$$

$$\therefore y = \mathrm{C} + \int (1{\cdot}2x^5 - 0{\cdot}5x)\,dx.$$

$$\therefore y = \mathrm{C} + 1{\cdot}2\int x^5\,dx - 0{\cdot}5\int x\,dx.$$

$$\therefore y = \mathrm{C} + 0{\cdot}2x^6 - 0{\cdot}25x^2. \quad \text{.......................(1)}$$

Now $y = 2{\cdot}3$ when $x = -1$.

Hence, substituting in (1),

$$2{\cdot}3 = \mathrm{C} + 0{\cdot}2 - 0{\cdot}25.$$

$$\therefore \mathrm{C} = 2{\cdot}35. \quad \text{.......................(2)}$$

$$\therefore y = 2{\cdot}35 + 0{\cdot}2x^6 - 0{\cdot}25x^2, \text{ by (1) and (2).}$$

Hence the value of y when $x = 0$ is $y = 2{\cdot}35$.

Example 3. *A particle moves with uniform acceleration of* 8·37 *ft. per sec. per sec. Initially it starts* 1·5 *ft. from the origin with a velocity of* 2·2 *ft. per sec. Find its distance from the origin after t sec.*

Since the acceleration is uniform and equal to 8·37 ft. per sec. per sec.,

$$\therefore \frac{dv}{dt} = 8{\cdot}37.$$

$$\therefore v = \mathrm{A} + \int 8{\cdot}37\,dt.$$

$$\therefore v = \mathrm{A} + 8{\cdot}37t. \quad \text{.......................(1)}$$

Now, initially (*i.e.* when $t = 0$) $v = 2{\cdot}2$. ....(2)

$$\therefore 2{\cdot}2 = \mathrm{A}, \text{ by (1) and (2).}$$

$$\therefore v = 2{\cdot}2 + 8{\cdot}37t. \quad \text{.......................(3)}$$

The equation (3) may as usual be written

$$\frac{ds}{dt} = 2{\cdot}2 + 8{\cdot}37t.$$

$$\therefore s = \mathrm{B} + \int 2{\cdot}2\,dt + \int 8{\cdot}37t\,dt.$$

$$\therefore s = \mathrm{B} + 2{\cdot}2t + 4{\cdot}185t^2. \quad \text{.......................(4)}$$

Now, initially (*i.e.* when $t = 0$) $s = 1{\cdot}5$. ....(5)

$$\therefore 1{\cdot}5 = \mathrm{B}, \text{ by (4) and (5).}$$

$$\therefore s = 1{\cdot}5 + 2{\cdot}2t + 4{\cdot}185t^2.$$

Example 4. *The pressure on the surface of a lake due to the atmosphere is known to be* 14 *lb. per sq. in. The pressure in the liquid x inches below the surface is known to be given by the law* $\frac{dp}{dx} = 0{\cdot}036$. *Find the pressure in the liquid at a depth of* 10 *feet.*

By the law of the pressure

$$\frac{dp}{dx}=0{\cdot}036.$$

$$\therefore\ p=\mathrm{C}+\int 0{\cdot}036dx.$$

$$\therefore\ p=\mathrm{C}+0{\cdot}036x. \quad\text{.................................}(1)$$

Now, at the surface of the lake (*i.e.* when $x=0$) $p=14$. ....(2)

$$\therefore \text{ by (1) and (2), } 14=\mathrm{C}.$$

$$\therefore\ p=14+0\ 036x. \quad\text{.................................}(3)$$

Hence the pressure at a depth of 10 feet (*i.e.* 120 in.) is given by

$$p=14+0{\cdot}036\times 120,$$

$$i.e.\ p=18{\cdot}3 \text{ lb. per sq. in.}$$

Example 5. *A particle is moving according to the law* $v=3{\cdot}1s^{\frac{1}{2}}$ (*where* v *ft. per sec. is its velocity and* s *ft. is its distance from the origin*). *Find the distance traversed after* 5 *sec., given that initially the particle is* 2 *ft. from the origin.*

By the given law of the motion

$$v=3{\cdot}1s^{\frac{1}{2}}.$$

$$\therefore\ \frac{ds}{dt}=3{\cdot}1s^{\frac{1}{2}}.$$

$$\therefore\ \frac{dt}{ds}=\frac{1}{3{\cdot}1}s^{-\frac{1}{2}}.$$

$$\therefore\ t=\mathrm{C}+\frac{1}{3{\cdot}1}\int s^{-\frac{1}{2}}\,ds.$$

$$\therefore\ t=\mathrm{C}+\frac{2}{3{\cdot}1}s^{\frac{1}{2}}. \quad\text{.....................................}(1)$$

Now, initially (*i.e.* when $t=0$), $s=2$. ....(2)

$$\therefore\ 0=\mathrm{C}+\frac{2\sqrt{2}}{3{\cdot}1}.$$

$$\therefore\ \mathrm{C}=-0{\cdot}91 \text{ (approx.).} \quad\text{..............................}(3)$$

$$\therefore\ t=\frac{2}{3{\cdot}1}s^{\frac{1}{2}}-0{\cdot}91. \quad\text{.....................................}(4)$$

$$\therefore\ s^{\frac{1}{2}}=1{\cdot}55\,(t+0{\cdot}91).$$

$$\therefore\ s^{\frac{1}{2}}=1{\cdot}55\times 5{\cdot}91, \text{ after 5 sec.}$$

$$\therefore\ s=84 \text{ ft. (very approx.).}$$

EXAMPLES VI.

1. Prove that $y=x^2-7$ is a solution of the differential equation

$$\frac{dy}{dx}=2x.$$

2. Prove that $y=x^2-7$ is a solution of the differential equation

$$\frac{d^2y}{dx^2}=2.$$

3. Prove that $y=x^2+C$ is a solution of both the differential equations

$$\frac{dy}{dx}=2x \quad \text{and} \quad \frac{d^2y}{dx^2}=2$$

for all numerical values of C.

4. Prove that $y=3x^4-5x^3+11x^2-4x$ is a solution of the differential equation

$$\frac{dy}{dx}=12x^3-15x^2+22x-4.$$

5. Find C so that the solution $x=1$, $y=3$ may satisfy the equation

$$y=C-x^2+x.$$

6. Find C in order that y may have the value 4 when $x=0$ in the equation

$$y=3x^4-5x^2+7x+C.$$

7. Find C in order that y may have the value 1·2 when $x=0·5$ in the equation

$$y=3x-\frac{4}{x}+C.$$

8. Express the following as differential coefficients,

$$\left[e.g.\ x^3=\frac{d}{dx}\left(\frac{x^4}{4}\right)\right]:$$

(i) x^2. (ii) x^5. (iii) $3x^2$. (iv) $4x^2$.
(v) $7x^5$. (vi) $3x^8$. (vii) x^2+1. (viii) $3x^2+x$.
(ix) $4x^3+3x$. (x) $2(x^2+x+1)$. (xi) x^{-2}. (xii) x^{-3}.
(xiii) $4x^{-2}-5x^{-3}$. (xiv) $5x^2-4x^{-2}$. (xv) $(x+3)^4$.
(xvi) $(3+4x)^3$. (xvii) $(5x-7)^4$. (xviii) $(x+1)^{\frac{4}{5}}$.
(xix) $(x+3)^{-\frac{1}{3}}$. (xx) $(x-7)^{-\frac{1}{2}}$. (xxi) $2(x-7)^{-\frac{2}{3}}$.

9. Find the following indefinite integrals:

(i) $\int x^4\,dx$. (ii) $\int x^9\,dx$. (iii) $\int x^{1·3}\,dx$.
(iv) $\int x^{0·2}\,dx$. (v) $\int x^{-2}\,dx$. (vi) $\int x^{-7}\,dx$.

(vii) $\int x^{-2\cdot7}\,dx.$ (viii) $\int x^{\frac{5}{3}}\,dx.$ (ix) $\int x^{\frac{7}{2}}\,dx.$

(x) $\int x^{-\frac{4}{5}}\,dx.$ (xi) $\int x^{-2\frac{1}{2}}\,dx.$ (xii) $\int (x^4+x^5+x^6)\,dx.$

(xiii) $\int (x^{1\cdot3}+x^{2\cdot4}+x^{3\cdot5})\,dx.$ (xiv) $\int (2+3x+4x^2)\,dx.$

(xv) $\int (a+bx^{\frac{1}{2}})\,dx.$ (xvi) $\int (2x^{0\cdot3}+3x^{4\cdot1}+5x^{7\cdot2})\,dx.$

✓ (xvii) $\int \frac{dx}{x^4}.$ (xviii) $\int \frac{dx}{x^9}.$

(xix) $\int \frac{x+1}{x^3}\,dx.$ ✓ (xx) $\int \frac{1+x+x^2}{x^7}\,dx.$

10. Find in their most general forms the solutions of the differential equations:

(i) $\frac{dy}{dx}=6x^8.$ (ii) $\frac{dy}{dx}=3x^2-5x^6+11x^7.$

(iii) $\frac{dy}{dx}=2x^{\frac{1}{2}}+3x-5x^{\frac{3}{2}}.$ (iv) $5\frac{dy}{dx}+x^{1\cdot32}=0.$

11. Solve $\frac{dy}{dx}=1+x$, given that $y=0$ when $x=1$.

12. Solve $3\frac{dy}{dx}=x^2-5$, given that $y=1$ when $x=2$.

13. Given $\frac{ds}{dt}-32t$ and $s=2$ when $t=0$, find the relation between s and t.

14. Given $\frac{ds}{dt}=3\cdot1+2\cdot3t$, find the relation between s and t, given that $s=0$ when $t=0$.

15. Given $\frac{dy}{dx}=x^2+1$ and $y=1$ when $x=2$, find the values of y when $x=0$, -1, 3.

16. Fill in the gaps in the following table, given that $\frac{dy}{dx}=3x+4$ and $y=-3$ when $x=1$:

x	0	0·5	1·3	2·1	−1·8	−0·7	−0·2
y							

17. Given $\frac{ds}{dt}=8s^{\frac{1}{2}}$, find the relation between s and t, given that $s=3$ when $t=1$. [Hint: Write the equation in the form $\frac{dt}{ds}=\frac{1}{8}s^{-\frac{1}{2}}$.]

18. A particle rolls down a smooth inclined plane according to the law $v=14{\cdot}2t$, where v is measured in feet per sec. and t in sec. If it starts from rest at the top of the plane, find how far it will have rolled down after 2 sec.

19. A particle is projected down the above inclined plane with an initial velocity so that the law of motion is known to be

$$v=7{\cdot}3+14{\cdot}2t,$$

the units being feet and seconds.

Find its distance down the plane after 2 seconds.

20. It is found that for a certain beam $1\frac{1}{2}$ feet long, fixed into the wall at one end and loaded at the other end with a weight, the deflection y inches at a distance x inches from the wall is given by $\frac{d^2y}{dx^2}=0{\cdot}0002(18-x)$. Find the law connecting y and x and the deflection at the extreme end.

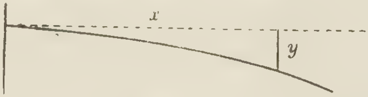


FIG. 42.

21. A flexible beam of wood ABC, 20 inches long, has a part AB, 2 inches long, inserted firmly in a wall so that AB is horizontal. A weight W is hung on at C and it is known that the deflection y inches from the horizontal, at a distance x inches from B, obeys the law

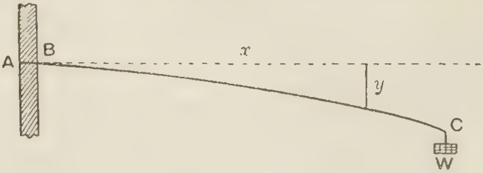


FIG. 43.

$$\frac{d^2y}{dx^2}=0{\cdot}002(18-x).$$

(i) Find $\frac{dy}{dx}$ at any point x.

(ii) Find the slope of the beam half-way along its length and at the end C, correct to the nearest minute. (Use your tables.)

(iii) Find the formula connecting y and x.

(iv) Find the deflection of the beam at its middle point and at its extremity C (correct to one place of decimals).

22. In the figure to last question it is known that the law of bending is $\frac{d^2y}{dx^2}=\mathrm{K}(18-x)$, where K is a constant depending on the material of the beam, thickness, etc., the beam being 20 inches long, and 2 inches being imbedded in the wall.

(i) Find $\frac{dy}{dx}$ at any point x.

(ii) Find the deflection y at any point x.

(iii) Find K from (ii), being given that the deflection at C is 3 inches.

(iv) Find the deflection half-way along the beam.

(v) Find the gradients half-way along the beam and at the end C (correct to minutes).

23. The slope of a curve is given by $\frac{dy}{dx}=2x-3$, and it passes through the point $(-1, -3)$. Find the equation to the curve, and sketch its graph roughly.

24. Find the equation to the curve passing through the origin, given by the differential equation $\frac{dy}{dx}=x^2+1$.

25. The figure represents a flexible ruler which is fixed vertically in a vice and plucked aside 1 inch. The end P is then let go, and the ruler vibrates according to the law $\frac{d^2x}{dt^2}=-0{\cdot}2x$ (which can be expressed in the form $\frac{d}{dx}\left(\frac{v^2}{2}\right)=-0{\cdot}2x$), where x is measured in inches and t in seconds. Find the speed with which P passes through the equilibrium position O. [*N.B.*—P may be considered for small displacements to describe a straight line.]

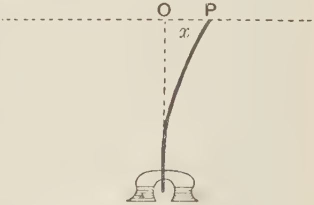


FIG. 44.

26. Find the equations to the curves, (i) through the origin, (ii) through the point (1, 1), (iii) through the point (0. −4), given by the differential equation $\frac{dy}{dx}=x$. Sketch the three curves roughly on squared paper.

27. Find the system of curves represented by the differential equation $\frac{dy}{dx}=2x^2+x$, and find the equations to those curves of the system that pass through (i) the origin, (ii) the point (0, −4), (iii) the point (0, 0·7), (iv) the point (1, 1). Sketch the four curves on squared paper.

28. If the law of motion of the end of the ruler in Qu. 25 is $\frac{d^2x}{dt^2}=-0{\cdot}08x$, and if the ruler be plucked aside 2 inches, find the velocity of its extremity when half-way between its extreme and mean positions.

$\left[\frac{d^2x}{dt^2}=-0{\cdot}08x\right.$ can be expressed in the form $\left.\frac{d}{dx}(\frac{1}{2}v^2)=-0{\cdot}08x.\right]$

29. If the law of motion of the extremity of the ruler in Qu. 25 is $\frac{d^2x}{dt^2}=-n^2x$, and if the ruler be plucked aside two inches and have a velocity of $\frac{1}{4}$ inch per second when passing through the mean position O, find n.

30. It is known that the law of motion of a bullet penetrating a sand-bank is $m\frac{d^2x}{dt^2}=-32\text{R}$, where m pounds is the mass of the bullet, R lb. wt. is the resistance of the sand, x feet is the distance penetrated, and t seconds is the time elapsed since the moment of first penetration. It is observed that a bullet of mass 3 oz. entering with a velocity of 2000 ft. per sec. penetrates a distance of $2\frac{1}{2}$ feet before coming to rest. Calculate R. [*N.B.*—It is assumed that the resistance offered to the passage of the bullet by the sand is sensibly constant during the motion.]

31. Two nails A and B are driven into a polished table, and a flexible wire is stretched very tightly and tied at A and B. A particle of mass m lb. is fastened to the mid-point P of the string, which is then pulled aside. If AB $=2a$ feet, it is known that the law of vibration is $m\frac{d^2x}{dt^2}+\frac{64\text{T}}{a}x=0$ where T lb. weight is the tension in the string. If the weight of the particle is 3 oz., the tension in the string 14 lb. weight, the distance between A and B one foot, find the velocity with which the particle passes through its mean position if it be pulled aside one inch to start with.

32. If a weight of T lb. pull down the pointer of a spring balance through x feet and the total work done is U ft.-lb., then it is known that $\frac{dU}{dx} = T = kx$ where k is a constant. If the pointer indicates 14 lb. when it has moved through 6 inches, what is the total work done in pulling the pointer (i) through 6 inches, (ii) through 8 inches, (iii) through 9 inches ?

33. If P ergs is the potential of a point on the axis of a short magnet distant r cm. from the middle of the magnet and F dynes is the intensity of the field there, then it is known that $-\frac{dP}{dr} = F = \frac{k}{r^3}$ where k is a constant depending upon the strength of the magnet. What is the potential of points 20, 30, 100 cm. from the middle point of a magnet whose k is 1000, the potential a great distance away being zero ?

34. If P ergs is the potential of a point r cm. from the centre of a charged sphere and F dynes the intensity of the field there, then it is known that $-\frac{dP}{dr} = F = \frac{Q}{r^2}$ where Q is a constant. If Q is 200 in the case of a sphere of radius 5 cm., what is the potential of points 10 cm. and 50 cm. from its centre, and what is the potential at the surface of the sphere itself ? [The potential a great distance away is zero.]

35. If R ohms is the resistance of a silver wire at $t°$ Centigrade, then it is known that $\frac{dR}{dt} = 0{\cdot}00377 \times R_0$ where R_0 ohms is the resistance at the temperature of melting ice. What is the resistance at 150° C. of a silver wire which has a resistance of 5 ohms at the temperature of melting ice ?

36. Using the notation of the preceding question,

$$\frac{dR}{dt} = R_0(0{\cdot}000{,}887{,}8 + 0{\cdot}000{,}002{,}074t)$$

in the case of mercury. If a certain column of mercury has a resistance of 7 ohms at 0° Centigrade, what is its resistance at 300° Centigrade ?

CHAPTER VII.

FURTHER USES OF SQUARED PAPER. AREAS AND THEIR INTERPRETATIONS.

1. We shall now show how to calculate approximately the area of a figure drawn on squared paper.

Suppose that we are required to find approximately the area of OAHV (Fig. 45). We can proceed in various ways. Thus, we might count the squares, making allowance for those squares through which the curve goes. This is, however, rather long and laborious.

A convenient method is to take the ordinates parallel to the y-axis at suitable intervals, to regard the curve between two consecutive ordinates as a straight line, and hence to calculate the areas of the resulting trapezia.

Hence we obtain :

Area of $\mathrm{O}a a\mathrm{V} = \mathrm{O}a \times p\mathrm{P} = 1 \times 1{\cdot}61 = 1{\cdot}61$ sq. in.

Area of $\alpha\beta ba = \alpha\beta \times qQ = 1 \times 2{\cdot}07 = 2{\cdot}07$ sq. in.

Area of $\beta\gamma cb = \beta\gamma \times r\mathrm{R} = 1 \times 2{\cdot}85 = 2{\cdot}85$ sq. in.

Area of $\gamma\delta dc = \gamma\delta \times s\mathrm{S} = 1 \times 3{\cdot}79 = 3{\cdot}79$ sq. in.

Area of $\delta\varepsilon ed = \delta\varepsilon \times t\mathrm{T} = 1 \times 4{\cdot}90 = 4{\cdot}90$ sq. in.

Area of $\varepsilon\mathrm{AH}e = \varepsilon\mathrm{A} \times u\mathrm{U} = 1 \times 6{\cdot}44 = 6{\cdot}44$ sq. in.

Hence total area $= 21{\cdot}66$ sq. ins. very approx.

Again, it is plain that a more accurate value for the area can be found by taking the ordinates closer. Thus, let us find the area of Fig. 45 by taking ordinates at intervals of half an inch. As the mean ordinates are not marked, we must use the formula "area of trapezium = half the sum of the parallel sides multiplied by the perpendicular distance between them."

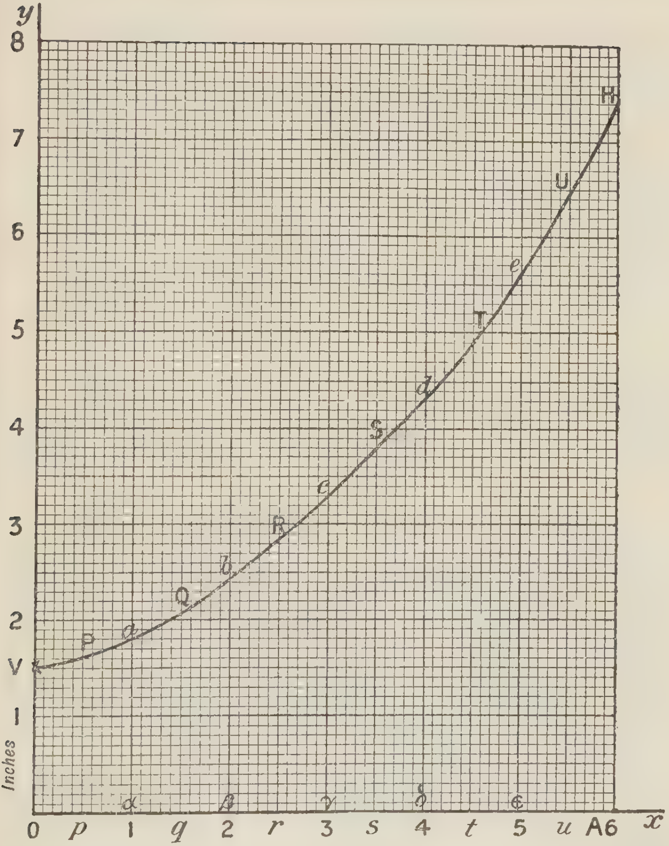


FIG. 45.

$$\text{Area of } \mathrm{O}p\mathrm{PV} = \frac{\mathrm{O}p}{2}(\mathrm{OV}+p\mathrm{P}) \quad = 0{\cdot}25(1{\cdot}50+1{\cdot}61) \text{ sq. in.}$$

$$\text{Area of } p\alpha a\mathrm{P} = \frac{p\alpha}{2}(p\mathrm{P}+\alpha a) \quad = 0{\cdot}25(1{\cdot}61+1{\cdot}80) \text{ sq. in.}$$

$$\text{Area of } \alpha q\mathrm{Q}a = \frac{\alpha q}{2}(\alpha a+q\mathrm{Q}) \quad = 0{\cdot}25(1{\cdot}80+2{\cdot}07) \text{ sq. in.}$$

$$\text{Area of } q\beta BQ = \frac{q\beta}{2}(qQ + \beta b) = 0{\cdot}25(2{\cdot}07 + 2{\cdot}43) \text{ sq. in.}$$

$$\text{Area of } \beta rRb = \frac{\beta r}{2}(\beta b + rR) = 0{\cdot}25(2{\cdot}43 + 2{\cdot}85) \text{ sq. in.}$$

$$\text{Area of } r\gamma cR = \frac{r\gamma}{2}(rR + \gamma c) = 0{\cdot}25(2{\cdot}85 + 3{\cdot}30) \text{ sq. in.}$$

$$\text{Area of } \gamma sSc = \frac{\gamma s}{2}(\gamma c + sS) = 0{\cdot}25(3{\cdot}30 + 3{\cdot}79) \text{ sq. in}$$

$$\text{Area of } s\delta dS = \frac{s\delta}{2}(sS + \delta d) = 0{\cdot}25(3{\cdot}79 + 4{\cdot}31) \text{ sq. in.}$$

$$\text{Area of } \delta tTd = \frac{\delta t}{2}(\delta d + tT) = 0{\cdot}25(4{\cdot}31 + 4{\cdot}90) \text{ sq. in.}$$

$$\text{Area of } t\varepsilon eT = \frac{t\varepsilon}{2}(tT + \varepsilon e) = 0{\cdot}25(4{\cdot}90 + 5{\cdot}62) \text{ sq. in.}$$

$$\text{Area of } \varepsilon uUe = \frac{\varepsilon u}{2}(\varepsilon e + uU) = 0{\cdot}25(5{\cdot}62 + 6{\cdot}44) \text{ sq. in.}$$

$$\text{Area of } uAHU = \frac{uA}{2}(uU + AH) = 0{\cdot}25(6{\cdot}44 + 7{\cdot}40) \text{ sq. in.}$$

Hence, adding the areas and making obvious arithmetical simplifications, we obtain 21·79 sq. in.

N.B.—The closer we take the ordinates, the better an approximation shall we obtain to the true area of the figure.

Example 1. *A man takes soundings of a river* 30 *feet wide at intervals of* 3 *feet across. Tabulate on squared paper, and hence find as approximately as you can the area of the cross-section of the river's channel.* (*See Fig.* 46.)

Distance from bank in feet	0	3	6	9	12	15	18	21	24	27	30
Depth of sounding in feet	2	3	5	7	9	10	12	8	7	4	3

Treating the section as made up of 10 trapezia, we see that its area

$$\begin{aligned} &= \tfrac{1}{2}\times 3(2+3) + \tfrac{1}{2}\times 3(3+5) + \tfrac{1}{2}\times 3(5+7) + \tfrac{1}{2}\times 3(7+9) \\ &\quad + \tfrac{1}{2}\times 3(9+10) + \tfrac{1}{2}\times 3(10+12) + \tfrac{1}{2}\times 3(12+8) \\ &\quad + \tfrac{1}{2}\times 3(8+7) + \tfrac{1}{2}\times 3(7+4) + \tfrac{1}{2}\times 3(4+3) \text{ sq. ft.} \\ &= \tfrac{3}{2}\{5+8+12+16+19+22+20+15+11+7\} \text{ sq. ft.} \\ &= \tfrac{3}{2}\times 135 \text{ sq. ft.} = 202{\cdot}5 \text{ sq. ft.} \end{aligned}$$

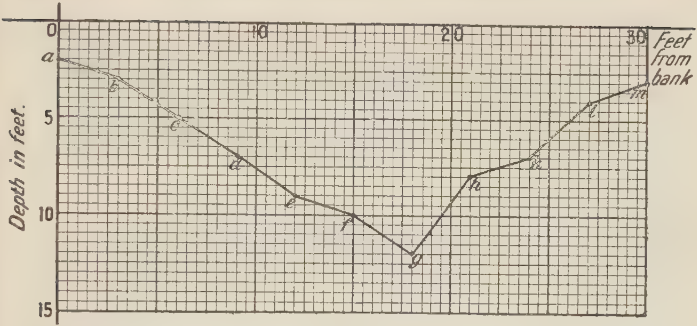


FIG. 46.

Example 2. *Trace as carefully as possible the graph of* $y = \frac{7x}{x+3}$ *from* $x = 0$ *to* $x = 10$. *Hence find as accurately as you can the area intercepted between the curve, the axis of* x *and the two ordinates corresponding to* $x = 0$ *and* $x = 10$.

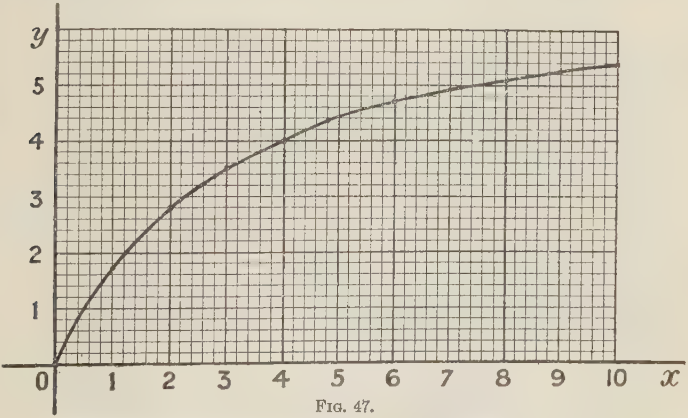


FIG. 47.

We construct the following table of coordinates:

x	0	1	2	3	4	5	6	7	8	9	10
y	0	1·75	2·8	3·5	4	4·38	4·67	4·9	5·09	5·25	5·38

and hence construct the above graph.

The required area is then

$$\begin{aligned}&= \tfrac{1}{2} \times 1\{(0+1{\cdot}75)+(1{\cdot}75+2{\cdot}8)+(2{\cdot}8+3{\cdot}5)+(3{\cdot}5+4)+(4+4{\cdot}38)\\ &\qquad +(4{\cdot}38+4{\cdot}67)+(4{\cdot}67+4{\cdot}9)+(4{\cdot}9+5{\cdot}09)+5{\cdot}09+5{\cdot}25)\\ &\qquad +(5{\cdot}25+5{\cdot}38)\}\text{ sq. units}\\ &= \{1{\cdot}75+2{\cdot}8+3{\cdot}5+4+4{\cdot}38+4{\cdot}67+4{\cdot}9+5{\cdot}09+5{\cdot}25+2{\cdot}69\}\\ &\qquad \text{sq. units}\\ &= 39{\cdot}03\text{ sq. units.}\end{aligned}$$

2. Practical application of areas.

Application I. *Given the velocity and time, to find the distance described.*

Suppose we have a train moving with a uniform speed of 40 miles an hour on a straight bit of railway. Let us plot the velocity against the time.

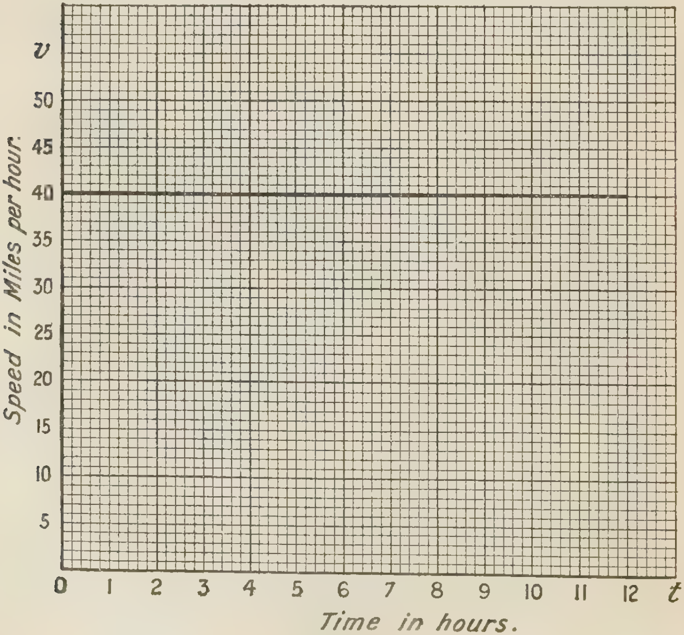


Fig. 48.

We obtain a line parallel to the horizontal axis and through the 40th division on the vertical axis.

Let us consider the space lying between the horizontal axis and the line parallel to it through the 40th division.

Now, in *one* hour, the distance described is 40×1 miles, and may therefore be represented graphically by the area enclosed between $t=0$ and $t=1$.

Also, in *two* hours, the distance traversed is 40×2 miles, and may therefore be represented graphically by the area enclosed between $t=0$ and $t=2$.

Again, in *three* hours, the distance traversed is 40×3 miles, and may therefore be represented graphically by the area enclosed between $t=0$ and $t=3$.

We therefore have the following general result :

If a body be moving with uniform velocity of v feet per second along a straight line, and if we plot v against t on squared paper, the area enclosed between the vertical axis, the vertical line representing the t^{th} *second of motion, the horizontal axis and the horizontal line representing the velocity of v ft. per sec., represents graphically the distance described by the body in t seconds of motion.*

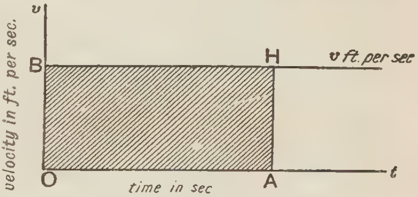


FIG. 49.

In the above diagram AH is the vertical ordinate representing the t^{th} second, and BH is the horizontal line representing graphically a constant velocity of v ft. per sec. The shaded area OAHB therefore represents graphically the total distance described.

Application II. *Body spinning with uniform velocity on an axis.*

Next consider the case of a body such as a wheel rotating with uniform velocity on an axis. Let the velocity of rotation be 60° per second. Then, if we plot the angular velocity ω against the time t, we obtain a graph as shown (Fig. 50).

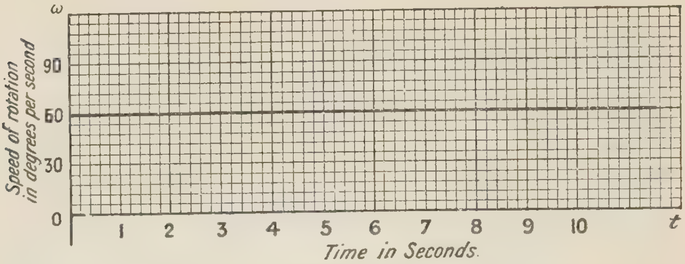


FIG. 50.

We obtain a line parallel to the horizontal axis through the 60th division on the vertical axis.

Let us consider the space enclosed between $\omega = 0$ and $\omega = 60$.

Now, in *one* second, the angle described is 60×1 degrees, and may therefore be represented graphically by the area enclosed between $t = 0$ and $t = 1$.

Also, in *two* seconds, the angle described is 60×2 degrees, and may therefore be represented graphically by the area enclosed between $t = 0$ and $t = 2$.

Again, in *three* seconds, the angle described is 60×3 degrees, and may therefore be represented graphically by the area enclosed between $t = 0$ and $t = 3$.

We therefore have the following general result :

If a body be rotating with a uniform angular velocity of ω degrees per second about an axle or pivot, and if we plot ω against

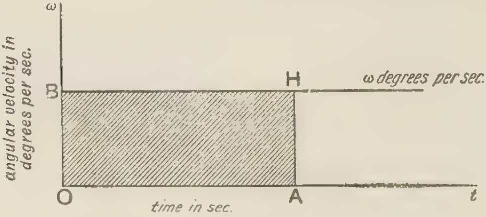


FIG. 51.

t on squared paper, the area enclosed between the vertical axis, the vertical line representing the t^{th} second of motion, the horizontal axis and the horizontal line representing the angular velocity

of ω degrees per second, represents graphically the total angle described by the body in t seconds of motion.

In the above diagram AH is the vertical ordinate representing the t^{th} second of motion, while BH is the horizontal line representing graphically a constant angular velocity of ω degrees per sec. The shaded area OAHB therefore represents graphically the total angle described by the rotating wheel.

Application III. There are many similar applications in Physics.

Force, Distance and Work.

When a boy pulls a box containing a stone along a rough pavement, he exerts a force by means of the string which is practically constant, and Physicists measure the energy which he has expended in pulling the box along by taking the product of the Force he exerts into the distance through which he has dragged the box. Physicists call this expenditure of energy the "Work Done," and they denote it by U. In fact, if F lb. weight be the constant force which the boy exerts and s feet be the distance through which he drags the box, then

$$U = F \times s,$$

the units of U being "foot-pounds" of work done, because the distance is estimated in "feet" and the force in "pounds."

If we therefore plot F against s, F being at present supposed constant, we obtain as graph a line parallel to the axis of x.

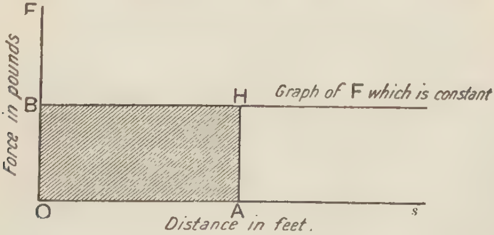


FIG. 52.

We therefore obtain a graph such as the above, and the shaded part OAHB represents the "work done," U by a constant force of F lb. dragging a body through a distance of s feet.

The same problem occurs in the case of water being raised from an old-fashioned draw-well by means of a bucket, and so on. Many such instances will occur to the student, *e.g.* horses drawing a plough or lorry, engine drawing a train, and so on.

The means of graphical representation is the same throughout.

3. Let us consider now the cases when v, ω and F are not uniform but vary.

We shall deal first of all with the case of a body moving in a straight line with a velocity which is not uniform, and let us plot v against t as before.

Suppose we obtain some such graph as that represented below :

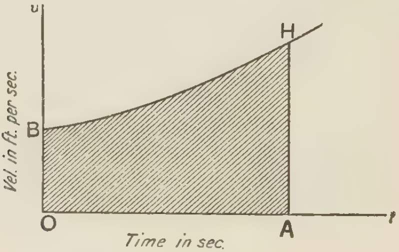


FIG. 53.

Then, judging *à priori*, we should expect the distance traversed in t sec. to be represented graphically on the above diagram by the area OAHB where OA represents t seconds.

Similarly, if we consider the case of an engine starting a train or a horse struggling to get a heavy railway truck into motion, a moment's consideration will show that neither the engine nor the horse exert a uniform force in getting the train and the truck respectively into motion. If we therefore plot F (the force exerted) against s (the distance traversed) we shall get a graph of the following form (Fig. 54).

[Since the force is biggest at the moment of starting a train or a truck, we should obtain a graph sloping downwards as shown.]

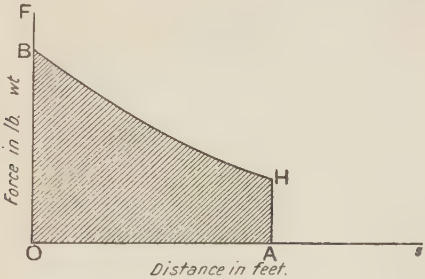


FIG. 54.

Just as before, judging *à priori*, we should expect the work done in moving the body through s feet to be represented on the above (s, F) graph by the area OAHB.

General Case. The student will be able to satisfy himself that this is the case from the following considerations. Consider the (t, v) diagram.

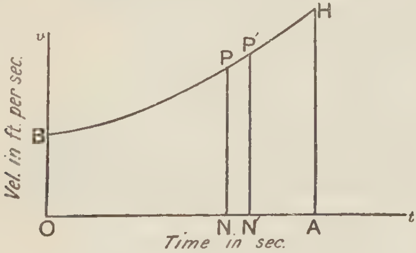


FIG. 55.

In describing the interval of time represented by NN′, the body has at the beginning of the interval a velocity represented by NP, and at the end a velocity represented by N′P′. The total distance described is therefore greater than if the body had a velocity represented by NP and less than if the body had a velocity represented by N′P′. We shall not be far "out," therefore, if we regard the space described as approximately the same as would be described by a body moving with a velocity the average of the above two velocities, viz.

$$\frac{\mathrm{NP}+\mathrm{N'P'}}{2},$$

provided that the interval of time represented by NN′ be small enough.

Hence the space described by the body during the interval represented by NN′ is very approximately represented by the area $\mathrm{NN'}\times\frac{\mathrm{NP}+\mathrm{N'P'}}{2}$, *i.e.* by the area of the trapezium NN′P′P in the figure (where the very short arc PP′ may be regarded as a straight line very approximately). It is therefore plain from this reasoning that the total distance described will be represented by the area OAHB, since the area of OAHB is the sum of the areas of the trapezia of which PNN′P′ is the type.

More rigorous reasoning than the above would carry us beyond the limits of the present book.

Example 1. *It is observed that the velocity of a body at the intervals of time noted is given by the following table:*

Time in sec. -	t	0	1	2	3	4	5	6	7	8	9	10
Vel. in ft. per sec.	v	3·0	3·6	4·1	5·5	5·9	6·4	7·0	7·6	8·1	8·5	9·1

Plot these on squared paper, and hence find as accurately as you can the distance (in feet) traversed. (*See Fig.* 56.)

The distance traversed is represented graphically by the area included between the curve, the axes and the ordinate $x=10$.

∴ Distance traversed

$$\begin{aligned}&=\tfrac{1}{2}\times 1\{(3+3{\cdot}6)+(3{\cdot}6+4{\cdot}1)+(4{\cdot}1+5{\cdot}5)+(5{\cdot}5+5{\cdot}9)\\&\qquad+(5{\cdot}9+6{\cdot}4)+(6{\cdot}4+7{\cdot}0)+(7{\cdot}0+7{\cdot}6)+(7{\cdot}6+8{\cdot}1)\\&\qquad+(8{\cdot}1+8{\cdot}5)+(8{\cdot}5+9{\cdot}1)\}\text{ feet}\\&=\left\{\frac{3+9{\cdot}1}{2}+3{\cdot}6+4{\cdot}1+5{\cdot}5+5{\cdot}9+6{\cdot}4+7{\cdot}0+7{\cdot}6+8{\cdot}1\right.\\&\qquad\left.+8{\cdot}5\right\}\text{ feet}\\&=62{\cdot}8\text{ feet.}\end{aligned}$$

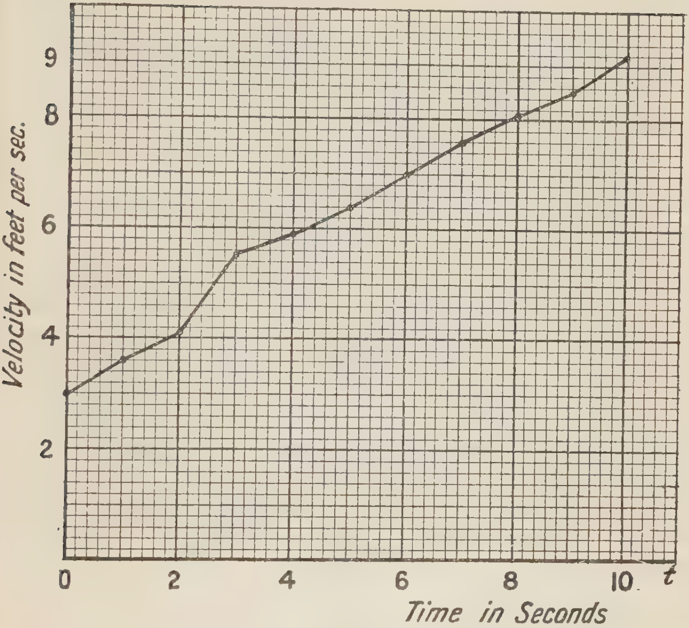


FIG. 56.

Example 2. *A force of* F *lb. wt. moves a body a distance of* s *feet in its own direction. Plot* F *against* s *from the subjoined data, and hence find as accurately as you can the total work done (in foot-lb.) in moving the body through* 10 *feet.*

s	0	1	2	3	4	5	6	7	8	9	10
F	100	105	115	130	120	110	95	83	72	71	70

The work done is represented graphically by the area included between the curve, the axes and the ordinate $x=10$.

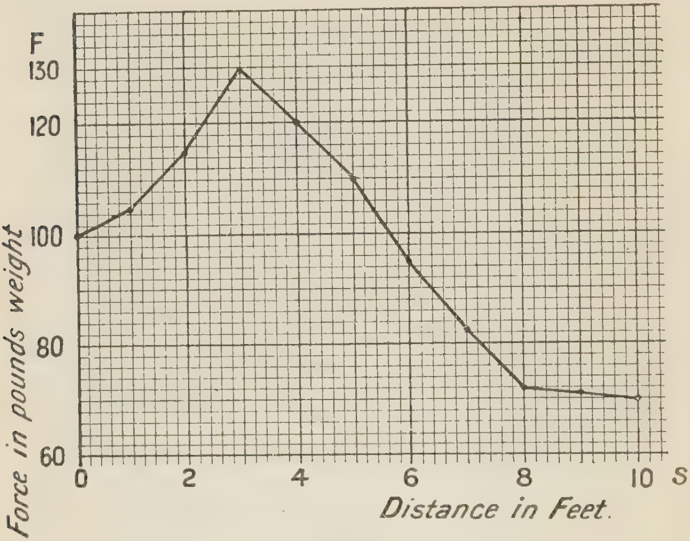


FIG. 57.

∴ Total work done

$$=\tfrac{1}{2}\times 1\{(100+105)+(105+115)+(115+130)+(130+120)$$
$$+(120+110)+(110+95)+(95+83)+(83+72)$$
$$+(72+71)+(71+70)\}\text{ ft.-lb.}$$
$$=\left\{\frac{100+70}{2}+105+115+130+120+110+95+83+72+71\right\}\text{ ft.-lb.}$$
$$=986\text{ ft.-lb.}$$

4. Notation for the approximate evaluation of the area.

Suppose we wish to find the area of the figure bounded by the curve AB, the axis of x, and the ordinates AH and BK parallel to the axis of y. We have already seen in Article 1 that we proceed by taking points N_1, N_2, N_3 ... conveniently near one another on the interval HK. We can then find *approximately* the area required by using the fact that,

$$\text{Area of AHKB} = \text{Sum of areas of } AHN_1P_1 + P_1N_1N_2P_2 + P_2N_2N_3P_3 + \ldots + P_7N_7KB \text{ (approx.).}$$

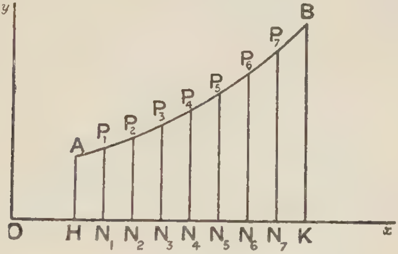


FIG. 58.

We can write this more briefly thus (see Fig. 59):

Let us regard the arc P′P″ as very approximately a straight line, since P′P″ is a very short arc. Then P′N′N″P″ is a representative small trapezium used in estimating *approximately* the area AHKB. Let PN be the mid-ordinate of the trapezium P′N′N″P″, *i.e.* the ordinate lying half-way between P′N′ and P″N″.

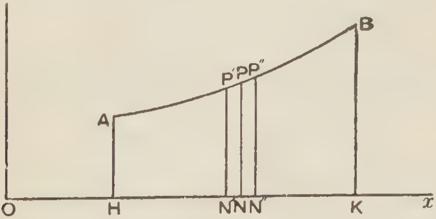


FIG. 59.

Then the area of P′N′N″P″ = PN × N′N″ (very approximately).

Hence, summing both sides,

Area of AHKB

= Sum of PN × N′N″ from H to K

= Σ(PN × N′N″) from H to K (more shortly written),

where Σ(PN × N′N″) stands for the "sum of" such small areas as PN × N′N″ throughout the interval from H to K, Σ being the Greek letter for S, the first letter of the word "Sum."

If we denote as usual PN by y and N′N″ by δx, we can denote the approximate evaluation by the methods of Art. 1 of the area AHKB by the formula $\Sigma y . \delta x$ in the interval from H to K.

Finally, if OH $= h$ and OK $= k$, we can denote still more shortly the approximate evaluation of the area AHKB by the methods of Art. 1 by the symbol

$$\sum_h^k y\,\delta x,$$

meaning that the approximation is carried out as x varies from the value h to the value k by conveniently selected increments denoted by δx.

5. Notation for the accurate evaluation of the area.

The student will now see that if the interval HK be further and further subdivided, the method described in Art. 3 will give a more and more accurate approximation to the value of the area. It will be plain, however, that it will be convenient to have a symbol representing the *accurate value* of the area. We do so by using an elongated S, viz. $\int$, to denote the "Limit of the Sum." We therefore have:

$$\textit{Approximate}\text{ value of area AHKB} = \sum_h^k y\,\delta x.$$

$$\textit{Accurate}\text{ value of area AHKB} = \int_h^k y\,dx.$$

N.B. -dx is used instead of δx in the accurate value of the area, because $\int_h^k y\,dx$ can no longer be regarded as obtained by dividing up HK into small intervals. $\int_h^k y\,dx$ is the value to which the chain of operations $\sum_h^k y\,\delta x$ tends to approximate as the subdivisions δx become more and more minute. Compare the relationship of $\frac{\delta y}{\delta x}$ and $\frac{dy}{dx}$ in the Differential Calculus.

6. Short method of evaluating an area approximately by dividing it up into trapezia.

In working out the examples we considered each trapezium separately, as we wished to emphasize the principle rather than to exhibit rapid methods of calculation. The actual work can, however, be carried out more rapidly and conveniently as follows:

Consider Fig. 58 on page 107.

Let $y_0 = AH$, $y_1 = P_1N_1$, $y_2 = P_2N_2$, ... $y_7 = P_7N_7$, $y_* = BK$.

Also let us suppose that all the subdivisions along the axis of x have been taken equal, so that $HN_1 = N_1N_2 = N_2N_3 = \text{etc.} = c$.

$$\therefore \text{ Approximate area} = \frac{y_0 + y_1}{2} \cdot c + \frac{y_1 + y_2}{2} \cdot c + \dots + \frac{y_7 + y_*}{2} \cdot c$$

$$= \left(\frac{y_0 + y_*}{2} + y_1 + y_2 + y_3 + \dots + y_7\right)c.$$

Hence, *if an area be divided approximately into trapezia by equidistant ordinates, the approximate area is equal to the distance apart of the ordinates multiplied by the sum of the intermediate ordinates plus the mean of the first and last.*

This result will be found of great utility in practical graphical work.

EXAMPLES VII.

1. Plot the following points, and hence find as accurately as you can the area enclosed between the curve and the axis of x:

x	0	1	2	3	4	5	6	7	8	9	10
y	0·0	0·9	1·5	2·1	3·2	3·9	3·7	3·0	2·1	1·1	0·0

2. Plot the following points, and hence find as accurately as you can the area enclosed between the curve, the axis of x and the ordinates $x = 5$ and $x = 12$.

x	5	6	7	8	9	10	11	12
y	2·3	2·7	3·2	3·8	4·1	4·3	3·9	3·2

3. A man wishes to find approximately the area of the cross-section of the channel of a river 30 feet wide, at a certain point in its course. He takes soundings at intervals of 3 feet, and obtains the following results:

Distance from the bank, in feet	0	3	6	9	12	15	18	21	24	27	30
Depth of the sounding, in feet	2	5	9	12	16	18	15	13	9	5	3

Calculate approximately the area of the cross-section.

4. A boy wishes to find the area of a piece of cardboard which he possesses.

The cardboard has roughly the shape of Fig. 60, and is bounded by two parallel edges. He measures the distance between these two parallel edges and finds it to be 14 inches. He then rules 6 lines at distances of 2 inches from one another, and on measuring them finds their lengths to be (including the edges), reading from left to right:

5 in., 7 in., 9 in., 10 in., 11 in., 9 in., 8 in., 7 in.

What is the area of the cardboard?

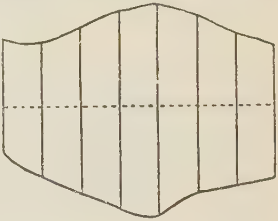


FIG. 60.

5. Draw the graphs of

x	0	10	20	30	40	50	60	70	80	90	100
y	12·1	13·2	14·0	15·0	15·9	16·5	16·0	15·4	14·3	12·1	10·2

and

x	0	10	20	30	40	50	60	70	80	90	100
y	12·1	8·4	6·5	4·2	2·9	0·7	0·9	2·3	4·7	7·3	10·2

and find as accurately as you can the area enclosed between them.

6. The following values of y and x being given, tabulate $\frac{\delta y}{\delta x}$ and $y \, . \, \delta x$ in each interval, and find A or the sum of such terms as

$y \, . \, \delta x$. Of course, A is the approximate area of the curve whose ordinate is y.

y	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
x	0	0·1736	0·3420	0·5000	0·6428	0·7660	0·8660	0·9397	0·9848	1·0000

(Bd. of Educ.)

7. An area is divided into ten equal parts by eleven equidistant parallel lines 0·2 in. apart, the first and last touching the bounding curve; the lengths of these lines or ordinates or breadths are, in inches:

0, 1·24, 2·37, 4·10, 5·28, 4·76, 4·60, 4·36, 2·45, 1·62, 0.

Find the area in square inches. (Bd. of Educ.)

8. The following values of y and x being given, tabulate $\frac{\delta y}{\delta x}$ and $y \, . \, \delta x$ in each interval. If $y \, . \, \delta x$ be called δA, find the value of A when $x = 0{\cdot}9$ if A is 0 when $x = 0$.

x	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
y	1·428	1·561	1·691	1·820	1·947	2·071	2·193	2·314	2·431	2·547

To facilitate tabulation, it will be found convenient to change these rows into columns. (Adapted from Bd. of Educ.)

9. x chains is the distance measured along a straight line AB from the point A; the values of y are offsets or distances in chains measured at right angles to AB to the border of a field. Find approximately the area of the field.

x	0	1·50	3·00	5·00	7·50	9·00
y	0·53	0·47	0·40	0·42	0·46	0·52

(Adapted from Bd. of Educ.)

10. The following numbers give v the speed of a train in miles per hour at the time t hours since leaving a railway station. In each interval of time, what is the distance passed over by the train?

v	0	2·4	4·7	7·2	9·6	12·0	14·3	16·9	18·9	20·7	22·2	23·4	24·3	24·9
t	0·0	0·4	0·8	1·2	1·6	2·0	2·4	2·8	3·2	3·6	4·0	4·4	4·8	5·2

At each of the times tabulated, what is x, the distance from the station? Tabulate your answers. (Bd. of Educ.)

11. Evaluate $\int_0^{10} y\,dx$ as accurately as you can from the following data, by drawing a graph on squared paper and estimating the area enclosed:

x	0	1	2	3	4	5	6	7	8	9	10
y	3	3·5	3·7	3·8	3·9	3·9	4·0	4·0	3·7	3·3	2·9

12. Evaluate by graphical methods $\int_0^{8} y\,dx$ as accurately as you can:

x	0	1	2	3	4	5	6	7	8
y	0	2	4	7	10	11	9	6	3

13. Evaluate by graphical methods $\int_0^{7} (y_1 - y_2)\,dx$ as accurately as you can from the following data, drawing the graphs of both y_1 and y_2:

x	0	1	2	3	4	5	6	7
y_1	3·0	3·9	4·5	5·8	6·5	6·9	7·4	8·1
y_2	0·0	0·7	1·2	1·7	2·1	2·5	2·9	3·4

14. A car whose mass is 2000 lb. starts from rest; the resistance to the motion is equal to 50 lb. wt. When it has travelled S feet, the force exerted by the engine is P lb. wt. where

S	0	10	20	30	40	50	60	70	80	90	100
P	644	634	622	607	587	565	537	509	475	440	404

Find, approximately, the velocity after the car has travelled 100 feet, assuming that the acceleration due to gravity is 32 in foot-second units. (Math. Trip.)

[Hint.—$v^2 = 0{\cdot}032\int \mathrm{P}\,d\mathrm{S} - 160$.]

15. On the indicator-diagram of a gas engine the following are some readings of p pressure and v volume. Calculate the total amount of "work" done as the volume increases from 2·0 to 3·1, given that "work" done $=\int_{2\cdot 0}^{3\cdot 1} p\,dv$.

v	2·0	2·1	2·2	2·3	2·4	2·5	2·6	2·7	2·8	2·9	3·0	3·1
p	84·5	110	176	215	231	234	226	213	202	192	183	175

(Adapted from Bd. of Educ.)

16. The velocity of a train which starts from rest is given by the following table, the time being reckoned in minutes from the start and the speed in miles per hour.

t, min.	2	4	6	8	10	12	14	16	18	20
v, m.p.h.	10	18	25	29	32	20	11	5	2	at rest.

Estimate approximately the total distance run in the 20 minutes. (Math. Trip.)

17. The specific heat of water s (in joules) at temperature $t°$ being given by the following table:

t	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°
s	4·219	4·195	4·181	4·174	4·173	4·174	4·178	4·184	4·190	4·197	4·205

find the number of joules required to heat 1 gram of water from 0° to 100°. (Math. Trip.)

[Hint.—Evaluate $\int_0^{100} s\,dt$ approximately.]

18. A car weighing 6 tons starts from rest under the action of a force given by the following table:

t in seconds	0	2	4	6	8	10	12	14	16	18	20
F in lb. wt.	780	750	708	620	495	420	365	324	300	280	270

If the resistances to motion are equivalent to a constant force of 40 lb. per ton, draw the acceleration-time curve, and find the velocity of the car at the end of the time. (Qual. Papers, Mech. Sc. Trip.)

[Hint:—$v=\frac{1}{420}\int_0^{20}(\mathrm{F}-240)\,dt$.]

19. A truck starting from rest and weighing 15 tons is drawn along the level against a constant resistance of 30 pounds per ton. The draw-bar pull is found to vary with the distance travelled according to the following table :

Distance travelled in feet, s	0	10	20	30	40	50
Draw-bar pull in pounds weight, F	900	890	868	822	763	679

Find (i) the kinetic energy of the truck, (ii) the velocity of the truck, (iii) the work done by the force, when the truck has travelled the first 50 feet.

[Hint :—Kinetic energy of the truck $= \left\{ \int_0^{50} \mathrm{F}\,ds - 450 \times 50 \right\}$ ft.-lb.

$$= \frac{1}{2} \times \frac{15 \times 2240}{32} v^2 \text{ ft.-lb.}$$

Work done by the force $= \int_0^{50} \mathrm{F}\,ds$ ft.-lb. where v ft. per sec. is velocity of truck.]

20. The following results were obtained during a tensile test on a steel bar :

Load in tons	1	2	3	4	5	6	7	8	9	10
Elongation in ten-thousandths of an inch	12	24	35	47	61	75	88	102	160	260

Plot a curve passing smoothly through the points, and find the work done, in mils-tons, during the test.

(Qual. Papers, Mech. Sc. Trip.)

CHAPTER VIII.

AREAS ENCLOSED BY GRAPHS, AND DEFINITE INTEGRALS.

1. Differential equation giving area enclosed by curve. Consider the following diagram :

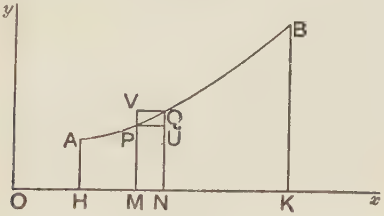


FIG. 61.

Let AB be the graph of $y=f(x)$ and let AH and BK be two ordinates such that

$$OH=a$$
$$OK=b.$$

We are required to find the area of the figure HKBA.

Let P and Q be two points on the curve conveniently near to one another. Let PM and QN be each perpendicular to OX. Draw PU perpendicular to QN and QV perpendicular to PM.

Let A denote the area of AHMP ;

δA denote the area of PMNQ ;

δx denote the length of MN.

We assume for convenience of exposition that the curve is sloping upwards from A to B.

Then plainly :

$$\text{Area of PMNU} < \text{area of PMNQ} < \text{area of VMNQ},$$

i.e. $$PM \times \delta x < \delta A < QN \times \delta x.$$

$$\therefore \; PM < \frac{\delta A}{\delta x} < QN.$$

Now let PM remain stationary while QN moves up indefinitely close to PM (by the method of repeated bisection of the interval MN or otherwise).

In the limit QN becomes indefinitely nearly equal to PM, *i.e.* y, while $\frac{\delta A}{\delta x}$ approaches its limiting value $\frac{dA}{dx}$.

Hence, since $\frac{\delta A}{\delta x}$ lies between y (*i.e.* PM) and something very nearly equal to y (viz. QN), we see that in the limit $\frac{dA}{dx} = y$.

By means of this differential equation, the required evaluations of areas can be made.

2. We give some examples.

Example 1. *Find the area enclosed between the curve $y = x^2$, the axis of x and an ordinate distant x from the origin.*

The curve and area to be dealt with are as follows :

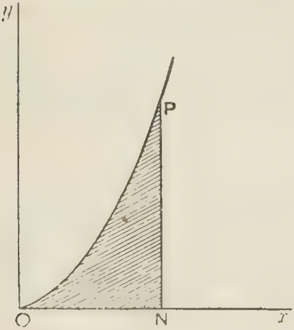


FIG. 62.

We have to find the area ONP where ON $= x$.

We have by the differential equation of Art. 1,

$$\frac{dA}{dx} = y.$$

$$\therefore \; \frac{dA}{dx} = x^2.$$

$$\therefore \; \frac{dA}{dx} = \frac{d}{dx}\left(\frac{x^3}{3}\right), \text{ by inspection.}$$

$$\therefore \; A = \frac{x^3}{3} + C. \quad \text{.....................(1)}$$

Now, if PN move along towards Oy we see that

$$A = 0 \quad \text{when} \quad x = 0. \quad \text{........(2)}$$

Hence substituting in (1), we find that C $= 0$.

$$\therefore \text{ the area of ONP is } \frac{x^3}{3}.$$

Example 2. *Find the area enclosed between the curve $y=x^2$, the axis of x, an ordinate distant 1 from the origin, and an ordinate distant x from the origin.*

The curve and area to be dealt with are as follows:

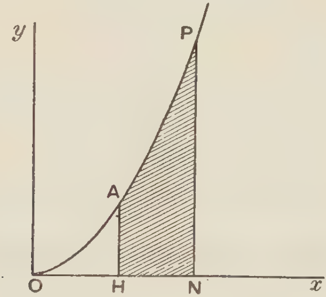


FIG. 63.

We have to find the area AHNP where $OH=1$, $ON=x$

We have by the differential equation of Art. 1:

$$\frac{dA}{dx}=y.$$

$$\therefore \frac{dA}{dx}=x^2.$$

$$\therefore \frac{dA}{dx}=\frac{d}{dx}\left(\frac{x^3}{3}\right), \text{ by inspection.}$$

$$\therefore A=\frac{x^3}{3}+C. \quad \text{.................................}(1)$$

Now, if PN move along till it approaches indefinitely close to AH, we see that the area enclosed between PN and AH vanishes.

Hence $A=0$ when $x=1$. ....(2)

Substituting from (2) in (1), we obtain $C=-\frac{1}{3}$.

$$\therefore A=\tfrac{1}{3}(x^3-1).$$

3. General method of calculating an area.

Consider the diagram of Art. 1.

Let the equation to the curve APQB be $y=f(x)$.

Let $OH=h$ and $OK=k$.

Let A be the area of AHMP.

$$\therefore \frac{dA}{dx} = y = f(x), \text{ by Art. (1).}$$

$$\therefore A = C + \int f(x)dx, \text{ by Chap. VI., Art. 8. } \dots\dots\dots (1)$$

Let $\int f(x)\,dx = \phi(x)$.

$$\therefore A = C + \phi(x). \dots\dots\dots (2)$$

When $x = h$, PM coincides with AH, and the area AHMP vanishes.

$$\therefore 0 = C + \phi(h), \text{ by (2). } \dots\dots\dots (3)$$

When $x = k$, PM coincides with BK and the area of AHKB is obtained.

$$\therefore \text{Area of AHKB} = C + \phi(k), \text{ by (2). } \dots\dots\dots (4)$$

Hence, by (3) and (4),

$$\text{Area of AHKB} = \phi(k) - \phi(h).$$

We agreed to denote the area of AHKB by $\int_h^k f(x)\,dx$ in Chap. VII., Art. 5.

Hence we have the general result :

$$\textit{Area of } \text{AHKB} \equiv \int_h^k f(x)\,dx = \phi(k) - \phi(h)$$

where $\phi(x) = \int f(x)\,dx$ *as defined in Chap.* VI., *Art.* 6.

Definition. $\int f(x)dx$ is called an "*indefinite integral.*"

$\int_h^k f(x)dx$ is called a "*definite integral.*"

Notation. $\phi(k) - \phi(h)$ is very conveniently written

$$\Big[\phi(x)\Big]_h^k,$$

and we shall use this notation throughout.

Example 1. *Evaluate* $\int_1^4 x^2 dx$.

Since $$\int x^2 dx = \frac{x^3}{3},$$

$$\int_1^4 x^2\,dx = \left[\frac{x^3}{3}\right]_1^4$$
$$= \left(\frac{4^3}{3} - \frac{1^3}{3}\right) = 21.$$

Example 2. *Evaluate* $\int_0^1 (1-x^2)\,dx.$

We have $$\int (1-x^2)\,dx = x - \frac{x^3}{3}.$$

$$\therefore \int_0^1 (1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_0^1$$
$$= (1-\tfrac{1}{3}) - (0-\tfrac{0}{3})$$
$$= \tfrac{2}{3}.$$

4. We shall now give some examples of the practical applications of Integration.

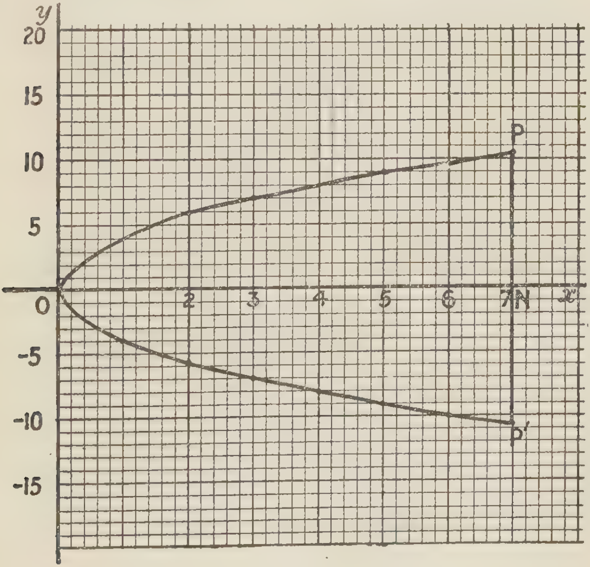


FIG. 64.

Example 1. *Find the area cut off from the parabola* $y^2=16x$ *by a double ordinate whose abscissa is* $x=7$.

We have to find $\int_0^7 2y\,dx$ (because x ranges from 0 to 7)

$$=\int_0^7 2\sqrt{16x}\,dx \quad (\because\ y=\sqrt{16x})$$

$$=8\int_0^7 x^{\frac{1}{2}}\,dx$$

$$=8\left[\tfrac{2}{3}x^{\frac{3}{2}}\right]_0^7$$

$$=\tfrac{16}{3}\left(7^{\frac{3}{2}}-0^{\frac{3}{2}}\right)=98{\cdot}8.$$

Example 2. *Find the area intercepted between the parabola* $y^2=16x$ *and the line* $y=x$.

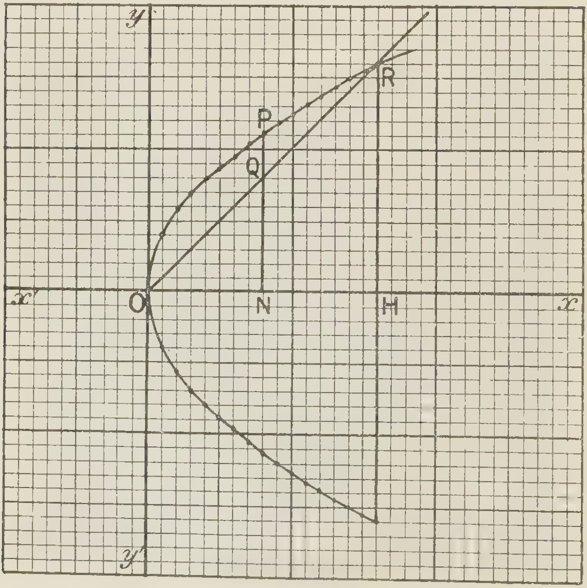


FIG. 65.

The figure explains itself.
Plainly $R \equiv (16, 16)$.

$$\text{Hence the area} = \int_0^{16} PQ\,dx \quad (\because x \text{ ranges from 0 to H})$$

$$= \int_0^{16} (4x^{\frac{1}{2}} - x)\,dx$$

$$= \left[\tfrac{8}{3}x^{\frac{3}{2}} - \frac{x^2}{2}\right]_0^{16}$$

$$= \tfrac{8}{3} \times 64 - 8 \times 16 = 42\tfrac{2}{3}.$$

Example 3. *Find the area enclosed between the parabolas* $y^2 = x$ *and* $x^2 = y$.

The figure is as follows :

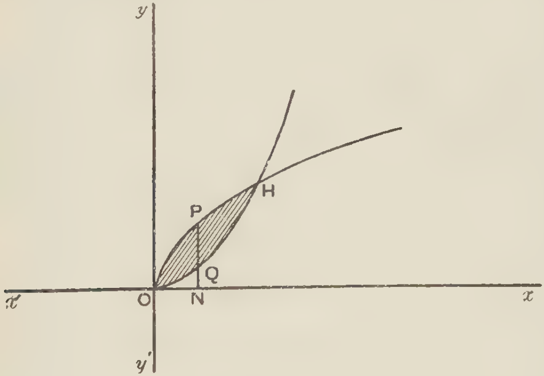


FIG. 66.

We obtain, on solving the equations $y^2 = x$ and $x^2 = y$ in the usual way, that

$$H \equiv (1, 1). \quad \text{...(1)}$$

$$\text{Hence the area required} = \int_0^1 PQ\,dx$$

$$= \int_0^1 (x^{\frac{1}{2}} - x^2)\,dx$$

$$= \left[\tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{1}{3}x^3\right]_0^1$$

$$= \tfrac{1}{3}.$$

Example 4. *The subjoined figure represents the cross-section of a trough of length 6 feet. If* OE = 1 *foot and* HK = $1\frac{1}{2}$ *feet, find the depth of water in the trough when it is half full* (*assuming that the curve* HOK *is a parabola*).

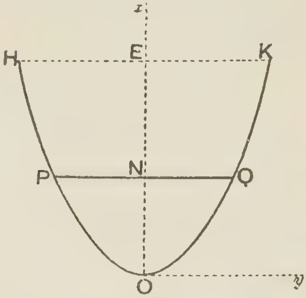


FIG. 67

Let D ft. be the depth of water in the trough, when it is half full.

Choose axes as shown on the figure, and assume that the equation to the parabola is

$$y^2 = 4ax.$$

∵ EK $= \frac{3}{4}$ ft. and OE = 1 ft. we take $y = \frac{3}{4}$ and $x = 1$ and insert in (1).

$$\therefore \frac{9}{16} = 4a.$$

$$\therefore a = \frac{9}{64}.$$

$$\therefore y^2 = \frac{9}{16}x.$$

$$\therefore y = \frac{3}{4}x^{\frac{1}{2}}.$$

Now the area of HPOQK $= \int_0^1 \text{PNQ}\,dx$

$$= \frac{3}{2}\int_0^1 x^{\frac{1}{2}}\,dx$$

$$= \frac{3}{2}\Big[\frac{2}{3}x^{\frac{3}{2}}\Big]_0^1$$

$= 1$ sq. ft. ....(2)

Also the area when half full $= \frac{3}{2}\int_0^{\text{D}} x^{\frac{1}{2}}\,dx$

$= \text{D}^{\frac{3}{2}}$ sq. ft. ....(3)

Hence we have to solve $\text{D}^{\frac{3}{2}} = \frac{1}{2}$,

i.e. D = 0·63.

Hence the water is 0·63 foot deep.

Example 5. *Draw the graph of* $y = 5x - x^2 - 4$ *and find the area enclosed by the portion of the curve above the axis of* x.

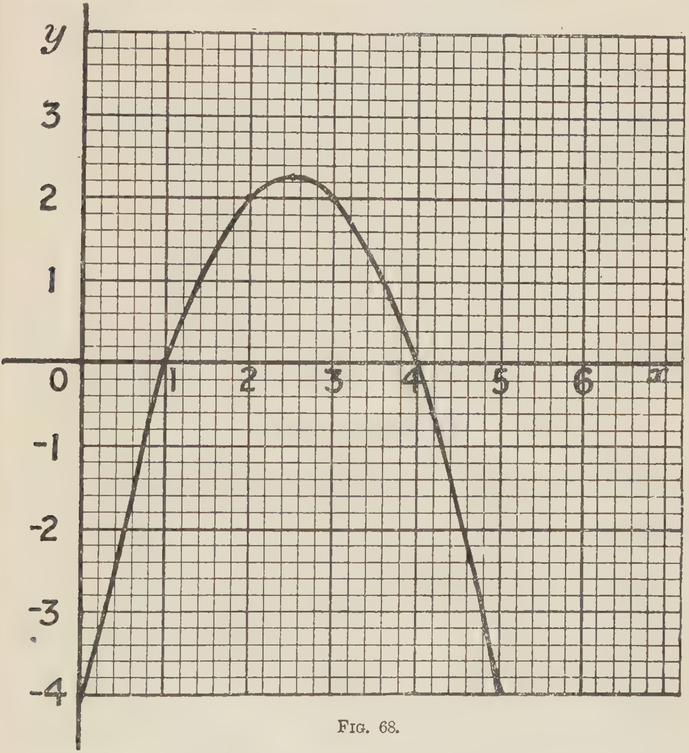


FIG. 68.

$$\text{Area required} = \int_1^4 (5x - x^2 - 4)\,dx = \left[\frac{5x^2}{2} - \frac{x^3}{3} - 4x\right]_1^4$$

$$= (40 - 21\tfrac{1}{3} - 16) - (2\tfrac{1}{2} - \tfrac{1}{3} - 4)$$

$$= 4{\cdot}5.$$

Example 6. *A quantity of gas expands so as to satisfy the law* $pv^{1\cdot 13} = 8000$. *Find the work done in expanding from* $v = 2$ *to* $v = 12$, p *being measured in pounds per square foot and* v *in cubic feet.*

N.B.—The work done by a gas when expanding is $\int p\,dv$.

Work done in ft.-lb.

$$= +\int_2^{12} 8000 v^{-1\cdot 13} dv = -\frac{8000}{0\cdot 13}\Big[v^{-0\cdot 13}\Big]_2^{12}$$

$$= \frac{8000}{0\cdot 13}\Big[\frac{1}{2^{0\cdot 13}} - \frac{1}{12^{0\cdot 13}}\Big]$$

$$= 11690 \text{ ft.-lb.}$$

Example 7. *A body projected from the earth vertically upwards with an initial velocity of* 100 *ft. per sec. is known to have its velocity given by the law* $v = 100 - 32t$ *where* t *is measured in seconds. Find the distance ascended in* 2 *seconds.*

N.B.—Distance traversed is given by $\int v\,dt$.

$$\text{Distance traversed} = \int_0^2 (100 - 32t)\,dt$$

$$= \Big[100t - 16t^2\Big]_0^2$$

$$= 136 \text{ feet.}$$

EXAMPLES VIII.

1. If A sq. in. is the area contained between the curve $y = 27x^3$, the axis of x and the straight line parallel to the axis of y and 2 in. from it, prove that $\frac{dA}{dx} = 27x^3$.

Hence find A when $x = 2$.

Draw a rough graph of the curve.

2. Draw a rough graph of the curve $y^2 = 9x^3$.

Prove that if A sq. cm. be cut off from it by a straight line parallel to axis of y and x cm. from it, then $\frac{dA}{dx} = 6x^{\frac{3}{2}}$.

Hence find A when $x = 4$.

3. Draw a rough graph of $y = 2x - x^2$. If A sq. cm. are intercepted between the curve, the axis of x and a parallel to the axis of y distant x cm. from it, prove that $\frac{dA}{dx} = 2x - x^2$. Hence find the area intercepted between the axis of x and the protion of the curve above the axis of x.

4. Roughly graph $yx^2 = 7$. A sq. cm. is the area intercepted between the axis of x, the curve and any two ordinates. Prove that $\frac{dA}{dx} = \frac{7}{x^2}$. Hence find the area intercepted between the ordinates distant 1 in. and 3 in. from the origin.

5. Find the value of :

(i) $\int_0^2 x^2\,dx$; (ii) $\int_1^3 x^3\,dx$; (iii) $\int_2^7 x\,dx$; (iv) $\int_1^5 (1+x^2)\,dx$;

(v) $\int_0^{2\cdot74} (1+x+x^2)\,dx$; (vi) $\int_0^{5\cdot4} x^{\frac{1}{2}}\,dx$; (vii) $\int_{1\cdot2}^{3\cdot7} \left(x^{-\frac{1}{2}}+x^{\frac{1}{2}}\right)dx$;

(viii) $\int_{2\cdot7}^{4\cdot2} (1+\sqrt{x})^2\,dx$; (ix) $\int_0^5 x^{4\cdot3}\,dx$; (x) $\int_2^6 x^{1\cdot3}\,dx$;

(xi) $\int_{2\cdot3}^{4\cdot7} x^{2\cdot2}\,dx$.

6. Sketch the graph of $y=x^2$, and hence find the areas included between the curve, the axis of x and the following ordinates :

(i) $x=0$ and $x=5$. (ii) $x=1$ and $x=3$.
(iii) $x=2$ and $x=12$. (iv) $x=1\cdot3$ and $x=6\cdot2$.

Tabulate your results.

7. Sketch the graph of $y=x^{\frac{1}{2}}$, and hence find the areas included between the curve, the axis of x and the following ordinates :

(i) $x=0$ and $x=11$. (ii) $x=1$ and $x=7$.
(iii) $x=2$ and $x=8$. (iv) $x=3\cdot7$ and $x=10\cdot5$.

Tabulate your results.

8. Find the area enclosed between the graph of $y=x^3+x+1$, the axis of x and the ordinates $x=1$, $x=6$.

9. Find the area enclosed between the graph of $y=x^{\frac{1}{2}}+x^{-\frac{1}{2}}$, the axis of x and the ordinates $x=5$ and $x=15$.

10. Sketch the graph of $y=x(4-x)$, and hence find the area of the space enclosed between the axis of x and the portion of the graph lying above the axis of x.

11. Sketch the graph of $y=x(10-x)$, and hence find the area of the space enclosed between Ox and the portion of the curve lying above Ox.

12. Sketch the graph of $y=(x-1)(10-x)$, and hence find the area of the space enclosed between Ox and the portion of the curve lying above Ox.

13. Sketch the graph of $y=(x-2)(14-x)$, and hence find the area of the space enclosed between Ox and the portion of the curve lying above the axis of x.

14. In the case of the parabola $y=x^{\frac{1}{2}}$, if P be a point on the curve and PM, PN the perpendiculars from P on Ox and Oy respectively, prove that the area enclosed between PM, the axis of x and the curve is equal to two-thirds the area of the rectangle OMPN.

15. In the case of the curve $y=x^n$, if P be a point on the curve and PM, PN the perpendiculars from P on Ox and Oy respectively, prove that the area enclosed between PM, the axis of x, and the curve is equal to $\frac{1}{n+1}$ of the area of the rectangle OMPN.

16. In the figure P is a point on the curve $y=x^2$. Prove that the curve divides the rectangle OMPN into two portions whose areas are in the ratio 2 : 1.

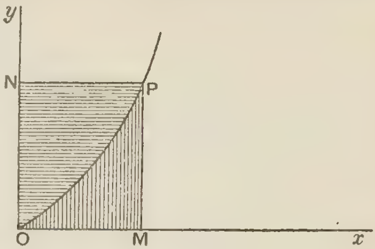


FIG. 69.

17. In the above figure to the preceding question, prove that if the curve be $y=x^n$, the areas are in the ratio of n : 1.

18. Prove that the line $y=x$ cuts the curve $y=x^3$ in the point (1, 1), and hence find the area enclosed between the line and the curve.

19. Prove that the line $y=x$ cuts the curve $y=x^n$ in the point (1, 1), and hence find the area enclosed between the line and the curve.

20. Prove that the curves $y=x^m$ and $y=x^n$ intersect in the point (1, 1), and hence find the area intercepted between the two curves.

21. Find the area enclosed between the curves $y=x^n$ and $y^n=x$.

22. If the velocity of a body be given by the law $v=10+3t$, calculate the distance traversed in the interval $t=1$ to $t=6$.

23. If the velocity of a body be given by $v=12-3t$, find when motion ceases, and hence calculate the total distance traversed from the beginning of the motion till the body comes to rest.

24. The work done by a gas in expanding is given by the formula $\int p\,dv$ where p is its pressure in pounds per sq. ft. and v is its volume in cu. feet. Express as a definite integral the work done by a gas in expanding (i) from 3 cu. ft. to 9 cu. ft., (ii) from v_1 cu. ft. to v_2 cu. ft. [*N.B.*—This definite integral can be evaluated when the law connecting p and v is known.]

25. The arch of a bridge is parabolic in form. It is 5 feet wide at the base and 5 feet high. Find the volume of water that passes through per second in a flood when the water is rushing at the rate of 10 feet per second.

26. A long trough whose cross-section is parabolic is $1\frac{1}{2}$ feet wide at the mouth and 2 feet deep. Find what depth of water there is in it when it is half-full.

27. Show that the area enclosed between the concentric circles of the diagram is less than the area of the rectangle whose length is the circumference BQM and breadth AB; and greater than the area of the rectangle whose length is APL and breadth AB. Hence show that if $OA=r$, $AB=\delta r$ and the area of the portion between the two circles is δA.

$$2\pi r<\frac{\delta A}{\delta r}<2\pi(r+\delta r).$$

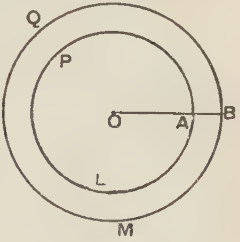


FIG. 70.

28. Deduce from the result of the preceding question that the area of a circle of radius r is given by the differential equation

$$\frac{dA}{dr}=2\pi r,$$

and hence show that $A=\pi r^2$.

29. Draw the graph of $y=\sqrt{1-x}$ from $x=0$ to $x=1$, and find the area of the space intercepted between the curve and the axes.

30. Find the area between the curve $y=\frac{1}{x^2}$ and the axis of from $x=1$ to $x=2{\cdot}3$.

31. *Simpson's Rule.*

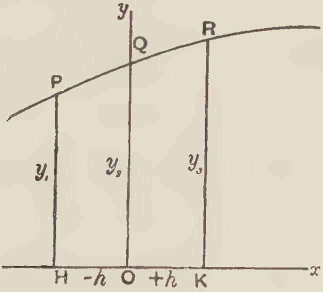


FIG. 71.

Let PQR be a portion of the arc of the curve $y=a+bx+cx^2$.

(i) Prove that $y_1=a-bh+ch^2$ (by inserting the coordinates of P).

(ii) Prove that $y_2=a$ (by inserting the coordinates of Q).

(iii) Prove that $y_3 = a + bh + ch^2$ (by inserting the coordinates of R).

(iv) Prove that $b = \frac{y_3 - y_1}{2h}$.

(v) Prove that $c = \frac{1}{h^2}\left(\frac{y_1 + y_3}{2} - y_2\right)$.

(vi) Prove that the area of PHKR is $2ah + \frac{2}{3}ch^3$.

(vii) Prove that the area of PHKR is $\frac{h}{3}(y_1 + y_3 + 4y_2)$.

[*N.B.*—This gives a convenient practical method of finding the *approximate* area enclosed by *any* given curve. We measure the extreme and mean ordinates, the distance between the extreme ordinates, and apply the above formula.]

(viii) By dividing the curve into a series of arcs which may be considered as approximately parabolic in shape, prove the following general Simpson's Rule :

Divide the area under consideration into an even number of strips of equal width. *The area is approximately "one-third the width of a strip multiplied by the sum of the first and last ordinates plus twice the sum of the remaining odd ordinates plus four times the sum of the even ordinates."*

[*N.B.*—This rule is of very wide use in the applications of practical mathematics.]

32. The work done in stretching an ordinary elastic string is $\int T\,dx$ foot-pounds, where T lb. wt. is the tension of the string and x feet is the extension beyond its natural length. It is found experimentally that in the case of the given string $T = 1{\cdot}3x$. Find the work done in stretching the string 6 inches beyond its natural length.

33. A certain quantity of gas at pressure p pounds weight and volume v cubic feet is encased in a cylinder with a movable piston so that no heat is allowed to enter or escape. It is then found experimentally that the (p, v) equation of the gas is $pv^{1{\cdot}3} = 27$. Find the work done by the gas in expanding from 2 to 3 cubic feet, given that the formula for the work done is $\int p\,dv$ foot-pounds.

34. The pointer of a spring balance descends on the scale a distance proportional to the weight hung from the hook. If 5 lb. pulls the pointer through 1 inch, (i) what is the total work done in moving the pointer through 6 inches from the zero position ; (ii) what is the energy stored up in the spring when a weight of 40 lb. is attached to the hook ?

35. A force of 20 tons compresses the spring buffer of a railway stop through 1 inch, and the force is always proportional to the compression produced. Find the work done by a train which compresses a pair of such stops through six inches.

36. The force of repulsion in dynes between two electric charges is given by multiplying together the number of units in the charges and dividing by the square of the number of centimetres distance between the charges. If a unit charge be brought from a very great distance to 10 cm. distance from a charge of 500 units, how many ergs of work are done? *Note.*—This is called the "potential" of the 500 units at a point 10 cm. away.

37. If a quantity of electricity is distributed uniformly over a sphere, its repulsive effect on the unit charge is the same as if it were concentrated at the centre of the sphere. If 700 units are upon a sphere of radius 10 cm., what is the potential of a point 30 cm. from the centre of the sphere, and what is the potential of a point on the sphere itself?

38. If Q units of electricity are upon a conductor whose capacity is C (a constant), and the potential of the conductor be V ergs, then it is known that $Q = CV$. Now to carry up an additional quantity dQ of electricity requires an expenditure of $V\,dQ$ ergs of work. What is the work required to be done to charge the conductor (initially uncharged) with 1000 units of electricity?

39. The attraction of the earth upon a body outside it is $\frac{M}{x^2}$, where M is a constant and x the distance of the body from the earth's centre. The attraction of the earth upon a 1 lb. mass at its surface (*i.e.* at 4000 miles from its centre) is the weight of 1 lb. What is the attraction on a 1 lb. mass at (i) 1000 miles from the surface, (ii) at 2000 miles from the surface, (iii) at x miles from the surface?

40. Calculate the work done in mile-pounds by the earth's attraction on a 1 lb. mass as it is drawn from 1100 miles from the surface to 1000 miles from the surface.

CHAPTER IX.

VOLUMES.

1. Approximate evaluation of a volume.

We shall begin this chapter by showing how to evaluate approximately by practical methods volumes that occur in every-day life.

Example 1. *A tree trunk* 12 *feet long is sawn up into blocks each* 6 *inches thick. The areas of the cross-sections of the blocks are found by measuring their diameters and using the formula* πr^2. *The areas thus estimated and calculated are given by the following diagram :*

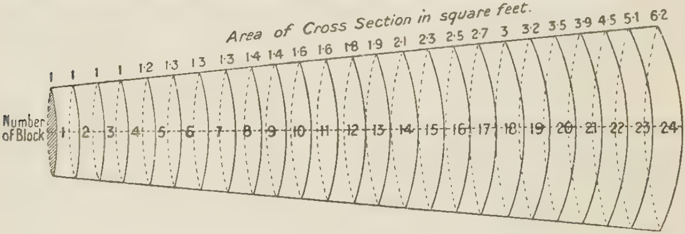


FIG. 72.

Calculate as accurately as you can the volume of the wood.

In applying an approximate process to the evaluation of the volume of a rough block bounded by two parallel faces of areas A and A′ sq. ft. respectively, it will obviously be sufficient if the block be not too thick to take the average of the areas of the two faces and multiply it by the thickness t feet, thus obtaining

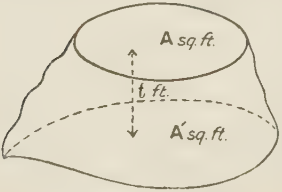


FIG. 73

$$\frac{A + A'}{2} \times t.$$

Applying this process to the above, we find that

the approx. volume of the first block is $\frac{1+1}{2} \times \frac{1}{2}$ cu. ft. $= 0{\cdot}5$ cu. ft.

,, ,, ,, second ,, $\frac{1+1}{2} \times \frac{1}{2}$ cu. ft. $= 0{\cdot}5$ cu. ft.

,, ,, ,, third ,, $\frac{1+1}{2} \times \frac{1}{2}$ cu. ft. $= 0{\cdot}5$ cu. ft.

,, ,, ,, fourth ,, $\frac{1+1{\cdot}2}{2} \times \frac{1}{2}$ cu. ft. $= 0{\cdot}55$ cu. ft.

,, ,, ,, fifth ,, $\frac{1{\cdot}2+1{\cdot}3}{2} \times \frac{1}{2}$ cu. ft. $= 0{\cdot}63$ cu. ft.

and so on.

Proceeding in this way and calculating block by block, we obtain the desired approximation to the volume.

This method, though simple to understand, is rather long and laborious to apply, and the calculation can be shortened and simplified as follows, *provided the thicknesses t ft. of all the blocks are the same* (which often happens in practical processes).

Let ${}_1A_2$ denote the area of the cross-section lying between the first and second blocks, and so on. Hence the volume

$$= \frac{A_1 + {}_1A_2}{2} . t + \frac{{}_1A_2 + {}_2A_3}{2} . t + \ldots + \frac{{}_{23}A_{24} + {}_{24}A}{2} . t \text{ cu. ft.}$$

$$= \left(\frac{A_1 + {}_1A_2}{2} + \frac{{}_1A_2 + {}_2A_3}{2} + \ldots + \frac{{}_{23}A_{24} + {}_{24}A}{2} \right) t \text{ cu. ft.}$$

$$= \left(\frac{A_1 + {}_{24}A}{2} + {}_1A_2 + {}_2A_3 + \ldots + {}_{23}A_{24} \right) t \text{ cu. ft.}$$

We thus obtain the following rule, which is of general application when the thicknesses of the blocks are all the same.

Practical Rule. *Take the average of the areas of the two end-sections and add this average to the sum of all the areas of the intermediate cross-sections. Multiply the result thus obtained by the common thickness of the blocks. The result gives very approximately (if the common thickness of the blocks be quite small) the volume required.*

Apply this rule to the case of the above log; we obtain as the average of the two end sections $\frac{1+6{\cdot}2}{2}$, *i.e* $3{\cdot}6$ sq. ft.

Hence, adding this result to the areas of the intermediate section, we obtain $54{\cdot}2$.

Hence the approximate volume is $54{\cdot}2 \times \frac{1}{2}$ cu. ft., *i.e.* $27{\cdot}1$ cu. ft.

Example 2. *The cross-section of a tree* (A *sq. inches*) *at distance* x *inches from one end is as follows :*

x	10	30	50	70	90	110	130	150
A	120	123	129	129	131	135	142	158

What is the volume of the sound wood of the tree in cubic inches, its total length being 160 *inches, but the first* 10 *inches being decayed at either end ?*

As the thickness of each block into which we may imagine the tree divided is the same, viz. 20 inches, we may apply the rule of Example 1.

Average area of the two end sections $=\frac{120+158}{2}=139$ sq. in.

Hence, adding 139 to the sum of the intermediate sections, we obtain 928 sq. in.

$\therefore$ Volume $=928\times 20$ cu. in. $=18560$ cu. in.

Example 3. A *is the area in square yards of the cross-section of a railway cutting at the distance* x *yards along the railway from a certain point :*

x	0	30	60	90	120	150
A	110	120	143	125	112	95

What is the volume in cubic yards, which must be removed from $x=0$ *to* $x=150$. (Bd. of Educ.)

Here again the thicknesses are the same, and hence we may apply the quick rule of Example 1.

Volume

$$=\left(\frac{110+95}{2}+120+143+125+112\right).\,30 \text{ cu. yd.}$$

$$=18075 \text{ cu. yd.}$$

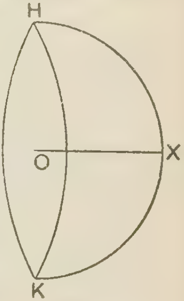


FIG. 74.

Example 4. *To find approximately by the practical method the volume of a hemisphere of radius* 4 *cm.*

Let OX be the radius about which the hemisphere is symmetrical.

Divide OX into (say) 8 equal parts, each of length half a cm.

Draw cross-sections through each of these points perpendicular to OX.

Then, calculating the areas of these cross-sections, we obtain the following chart, where A sq. cm. is the area of the cross-section distant x cm. from 0.

x cm.	0	0·5	1	1·5	2	2·5	3	3·5	4
A sq. cm.	16π	$15{\cdot}75\pi$	15π	$13{\cdot}75\pi$	12π	$9{\cdot}75\pi$	7π	$3{\cdot}75\pi$	0

Thus the volume of the hemisphere

$$=\left\{\frac{16+0}{2}+15{\cdot}75+15+13{\cdot}75+12+9{\cdot}75+7+3{\cdot}75\right\}\pi\times\tfrac{1}{2}\text{ c.c.}$$

$$=\{85\pi\times\tfrac{1}{2}\}\text{ c.c.}=42{\cdot}5\pi\text{ c.c.}$$

Now we know by other methods that

$$\text{Volume of the hemisphere}=\tfrac{2}{3}\pi\times4^3\text{ c.c.}=42{\cdot}67\pi\text{ c.c.}$$

Thus the approximative result is very fairly accurate.

2. Volume as a definite integral.

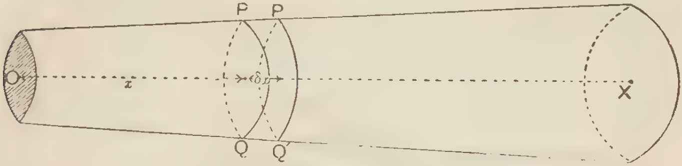


FIG. 75.

We have seen that if we wish to find approximately the volume of a body we divide it by convenient cross-sections, each perpendicular to a chosen axis X and of convenient thicknesses. Then, if we take a typical section such as that lying between PQ and P′Q′ above, the volume required is got by the formula

$$\Sigma(\text{A}\,.\,\delta x),$$

where Σ denotes "the sum of";

A is the arithmetic mean of the areas of the sections PQ and P′Q′;

δx denotes the thickness of the portion bounded by the cross-sections PQ and P′Q′.

Hence, if we require, *not an approximate value of the volume*, but its *actual accurate value*, we replace $\Sigma(\text{A}\,.\,\delta x)$ by the Definite Integral $\int_0^l \text{A}\,dx$, where l is the length of the body whose volume we require. A few examples will make the method of procedure plain.

Example 1. *Find the volume of a cone of semi-vertical angle* 45° *and of height* 12 *cm.*

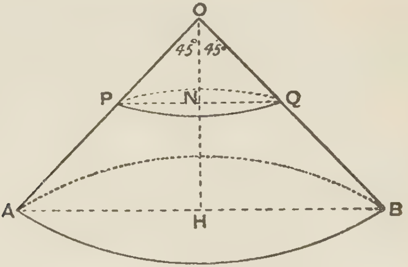


FIG. 76.

Let OH be the axis of the cone.
Let ON $=z$ cm. and let A sq. cm. be the area of the section PNQ.
Let V cu. cm. be the volume of the cone.

$$\therefore\ V=\int_0^{12} A\,dz \text{ c.c.}=\int_0^{12}(\pi\,.\,PN^2)\,dz \text{ c.c.}=\int_0^{12}\pi\,.\,z^2\,dz \text{ c.c.} \qquad (\because\ PN=ON=z)$$

$$=\pi\int_0^{12} z^2\,dz \text{ c.c.}=\pi\left[\frac{z^3}{3}\right]_0^{12}\text{c.c.}=1809{\cdot}5 \text{ c.c.}$$

Example 2. *Find the volume of a sphere.*

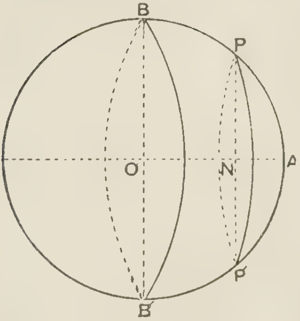


FIG. 77.

Take a radius OA and consider sections of the sphere perpendicular to this radius. Let A be the area of the section PNP′. Then, V denoting the volume of the sphere and a its radius,

$$V=2\int_0^a A\,dx \quad (x=ON)$$

$$=2\int_0^a(\pi\,.\,PN^2)\,dx$$

$$=2\pi\int_0^a(a^2-x^2)\,dx$$

$$=2\pi\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$=\tfrac{4}{3}\pi a^3.$$

Example 3. *Find the volume of a sphere of radius 4 cm., intercepted between two parallel sections whose perpendicular distances from the centre are 1 cm. and 2 cm. respectively (the sections being on the same side of the centre of the sphere).*

(For diagram, see the figure to last example.)

Then
$$V = \int_1^2 \pi . PN^2 . dx \text{ (where } x = ON)$$
$$= \pi \int_1^2 (16 - x^2)\, dx$$
$$= \pi \left[16x - \frac{x^3}{3} \right]_1^2$$
$$= \pi \times \tfrac{41}{3}$$
$$= 43{\cdot}0 \text{ cu. cm.}$$

Example 4. *Water is poured to a depth of D feet into a hemispherical pot of radius a ft. Find the volume of water contained.*

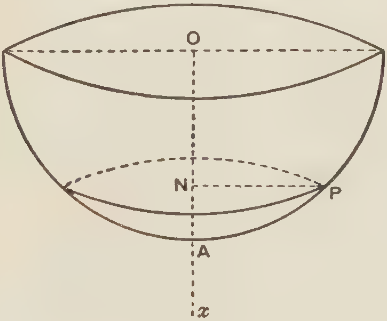


FIG. 78

As usual
$$V = \int_0^D \pi . PN^2 . dx$$
$$= \pi \int_0^D \{a^2 - (a - x)^2\}\, dx$$
$$= \pi \int_0^D (2ax - x^2)\, dx$$
$$= \pi \left[ax^2 - \frac{x^3}{3} \right]_0^D$$
$$= \pi \left(aD^2 - \frac{D^3}{3} \right) \text{ cu. ft.}$$

Example 5. *A circular stack of solid brickwork stands* 12 *feet high, gradually tapering from the base (whose diameter is* 4 *feet) to the top (whose diameter is* 2 *feet). Calculate the volume of brickwork it contains.*

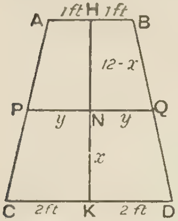


Fig. 79.

Let the above represent a section through the axis of symmetry HK of the figure.

Then plainly $\frac{2-y}{y-1}=\frac{x}{12-x}$ (as is easily seen if we draw a line through P parallel to HK and consider similar triangles).

Hence $y=2-\frac{1}{12}x$. ....(1)

Now the area of the cross-section of the stack by a plane through PQ parallel to the ground is πy^2.

Hence the volume $=\int_0^{12}\pi(2-\frac{1}{12}x)^2\,dx$

$$=\pi\left[4x-\tfrac{1}{6}x^2+\frac{x^3}{432}\right]_0^{12}$$

$$=\pi\times 28 \text{ cu. ft.}$$

3. To find the volume contained by a surface of revolution.

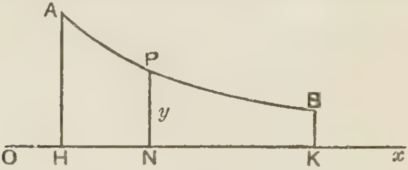


Fig. 80.

Let the above curve, which lies in the plane of the paper, rotate about Ox as axis, and so produce a surface of revolution. Then the area of the section generated by the ordinate PN is πy^2. Hence the volume contained by the surface of revolution thus generated is $\pi\int_h^k y^2\,dx$ where OH $=h$ and OK $=k$.

EXAMPLES IX.

1. A rough wooden block with parallel faces is 6 inches thick and has one face 2 sq. ft. and the other face 3 sq. ft. in area. Calculate *approximately* the volume.

2. A tree trunk 10 feet long has one end 2 sq. ft. in area and the other end $4\frac{1}{2}$ sq. ft. in area. The trunk tapers fairly regularly from its thick to its thin end. It is sawn into blocks each a foot long. The areas of the 9 sections made by the saw are approximately 2, 2, $2\frac{1}{2}$, $2\frac{1}{2}$, $2\frac{1}{2}$, 3, 3, 4, $4\frac{1}{2}$ square feet. Calculate approximately the volume of the trunk.

3. A man saws a circular log 4 ft. long into 5 portions of random length. His son out of curiosity measures the lengths of these 5 portions and also the diameters of the various cross-sections. The results he obtains are exhibited on the diagram:

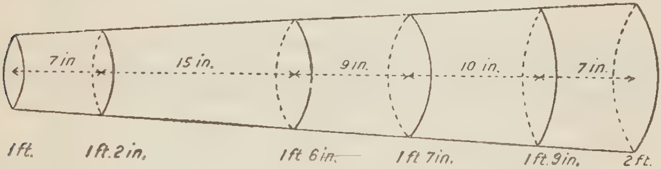


Fig. 81.

From these measurements he calculates approximately the volume. What approximate result will he obtain?

4. The following are the areas of cross-section of a body at right angles to its straight axis:

A square inches - -	250	292	310	273	215	180	135	120
x inches from one end -	0	22	41	70	84	102	130	145

Plot A against x on squared paper, and estimate approximately from your diagram the whole volume from $x=0$ to $x=145$.
(Bd. of Educ.)

5. In Qu. 4, if at $x=50$ a cross-sectional slice of small thickness δx has the volume δv, find $\frac{\delta v}{\delta x}$.
(Bd. of Educ.)

6. There is a natural reservoir with irregular sides. When filled with water to the vertical height h feet above the lowest point, the following is the area A of the water surface in thousands of square feet:

h	0	5	10	20	30	42	50	65	75
A	0	220	322	435	505	560	586	617	624

Plot A against h on squared paper, and hence find A when h is 36. (Bd. of Educ.)

7. Find the volume of water which would raise the surface from $h=35\frac{1}{2}$ to $h=36\frac{1}{2}$ in Qu. 6. (Bd. of Educ.)

8. If V cu. ft. be the volume of water in the reservoir of Qu. 6, show that $\delta V = A\,\delta h$ very approximately, and hence deduce that

$$\frac{dV}{dt} = A\frac{dh}{dt}.$$

9. A spring in the bottom of the reservoir of Qu. 6 emits water at the rate of 10 cu. ft. a minute. Find the rate at which the depth of water in the reservoir is increasing when the water is 10 ft. deep.

10. A cone of revolution has a semi-vertical angle of 45°. Find the volume included between two parallel planes perpendicular to the axis and distant 3 cm. and 7 cm. respectively from the vertex (using the formula vol. $=\int_3^7 A\,dx$).

11. Prove that in general the volume of a right-circular cone is $\frac{1}{3}$ (area of base) × (perpendicular height).

12. A cone of revolution has a semi-vertical angle α. Prove that the volume of the cone included between two parallel planes perpendicular to the axis and distant a cm. and b cm. respectively from the vertex is $\frac{\pi}{3}(a^3-b^3)\tan^2\alpha$ cu. cm.

13. Five cubic feet of water is run into a tank in the shape of a cone of revolution of semi-vertical angle 30°. Find the depth of water in the tank.

14. An ordinary bucket is 1 ft. 4 inches deep. The diameter of its bottom is 12 inches and the diameter of its mouth is 16 inches. Calculate its volume.

15. At a dairy-farm the milk plates are 3 feet across the top, $2\frac{1}{2}$ feet across the bottom and 4 inches deep. Find the volume of milk they can hold.

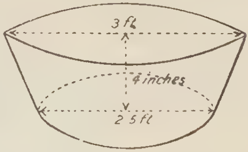


FIG. 82.

16. Find the volumes of the two parts of a sphere of radius 3 cm. into which it is divided by a plane whose perpendicular distance from the centre is 1 cm.

17. Find the volume of the frustum of a sphere of radius 3 ft. included between two parallel planes whose perpendicular distances from the centre are 1 ft. and 2 ft. respectively.

18. Water is poured into an upturned hemispherical pot of radius 2 ft. to a depth of $1\frac{1}{2}$ feet. Find by integration the volume of the water.

19. A cubic foot of water at ordinary temperatures weighs 62·3 lb. Find the weight of water in a hemispherical pot of radius 3 ft., the water being two feet deep.

20. An upturned hemispherical pot of radius 4 feet is filled with water to a depth x ft. Calculate the volume V in terms of x. Plot V against x on squared paper, and hence find the depths of water in the pot when (1) 20 cu. ft., (2) 30 cu. ft., (3) 45 cu. ft. of water have been emptied into the pot.

21. The parabola $y^2 = 8x$ is turned about its axis, thus generating a surface of revolution. Calculate the volume enclosed between this surface and a plane perpendicular to the axis and at distance 5 from the vertex.

22. A reel is formed by the revolution of the curve $x^2 = 25(y - 6)$ about the axis of x, and its two circular ends are each 5 from the origin. Find the volume of the reel.

23. A basin is formed by the revolution of the curve $x^3 - 64y$ about the axis of y. If the depth of the basin is 8 inches, how many cubic inches will it hold ?

24. A boring instrument is formed by the revolution of the curve $x^3 = 200y$ about the axis of x. What is the volume of the hole cut out when the instrument has penetrated 10 inches ?

CHAPTER X.

APPLICATION OF THE CALCULUS TO CENTRES OF GRAVITY, MOMENTS OF INERTIA AND CENTRES OF PRESSURE.

PART I. CENTRES OF GRAVITY.

1. It is proved in treatises on Statics that **if a system of particles of masses $m_1, m_2, m_3, \dots, m_n$ be placed at the points $(x_1, y_1), (x_2, y_3), (x_3, y_3), \dots, (x_n, y_n)$ respectively in the plane of the axes, the position of the Centre of Gravity (x, y) is given by the formulae :**

$$\bar{x}=\frac{m_1x_1+m_2x_2+\dots+m_nx_n}{m_1+m_2+\dots+m_n},$$

$$\bar{y}=\frac{m_1y_1+m_2y_2+\dots+m_ny_n}{m_1+m_2+\dots+m_n},$$

or, as it may be otherwise written,

$$\bar{x}=\frac{\Sigma mx}{\Sigma m}. \qquad \bar{y}=\frac{\Sigma my}{\Sigma m}.$$

Example. *A rectangular table, ABCD, has weights 4 lb., 9 lb., 7 lb. and 11 lb. respectively placed upon its four corners, A, B, C, D: AB being 8 ft. and AD 4 ft. Find the position of their Centre of Gravity.*

Let us calculate the distance of the centre of gravity G from AB.

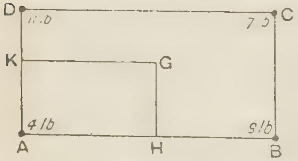


FIG. 83.

$(4+9+7+11) \cdot GH$
$=4\times 0+9\times 0+7\times 4+11\times 4.$

$\therefore\ GH=\frac{72}{31}=2{\cdot}32.$

Next let us calculate the distance of G from AD.

$(4+9+7+11) \cdot GK$
$=4\times 0+9\times 8+7\times 8+11\times 0.$

$\therefore\ GK=\frac{128}{31}=4{\cdot}13.$

Approximate Calculation of the Centre of Gravity of a continuous body.

If we have to deal with a continuous body of matter instead of separate particles, we regard the whole body as broken up into small masses δm, and we can then find approximately the position of the Centre of Gravity by using the modification of the above formula, namely :

$$\bar{x}=\frac{\Sigma x\,\delta m}{\Sigma\delta m}, \qquad \bar{y}=\frac{\Sigma y\,\delta m}{\Sigma\delta m},$$

where $\Sigma\delta m$ is obviously the total mass of the body.

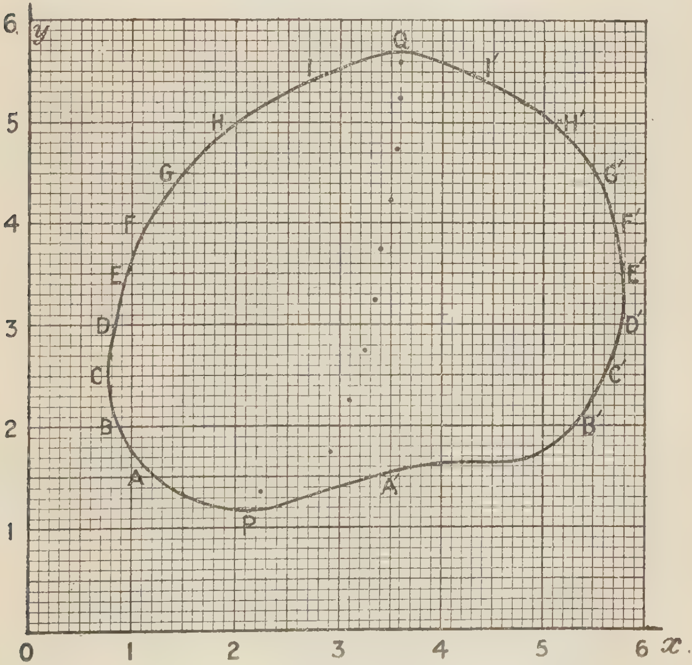


FIG. 84.

Example. *To calculate approximately the position of the centre of gravity of the above figure.*

Take two convenient axes Ox and Oy as shown.

Divide the figure into convenient sections by the lines AA′, BB′, CC′, ... II′, parallel to Ox.

Mark with a pencil the approximate positions (as estimated by the eye) of the Centres of Gravity of these various sections, and count the number of squares in a section (or otherwise estimate the area). Then apply the formulae as before to obtain the distance of the centre of gravity of the figure from Ox. Apply a similar procedure to Oy.

Name of section - -	PAA′	ABB′A′	BCC′B′	CDD′C′	DEE′D′
Area of section - -	41	183	233	245	244
Distance of centre of gravity of section from Ox as estimated by the eye	13·5	17·5	22·5	27·5	32·5
Distance of centre of gravity of section from Oy as estimated by the eye	22·5	29	31	32·5	33·5

Name of section - -	EFF′E′	FGG′F′	GHH′G′	HII′H′	IQI′
Area of section - -	235	216	178	102	12
Distance of centre of gravity of section from Ox as estimated by the eye	37·5	42·5	47·5	52·5	56
Distance of centre of gravity of section from Oy as estimated by the eye	34	35	35·5	36	36

Thus

$$\begin{aligned}\bar{y} = \{&41 \times 13{\cdot}5 + 183 \times 17{\cdot}5 + 233 \times 22{\cdot}5 + 245 \times 27{\cdot}5 + 244 \times 32{\cdot}5 \\ &+ 235 \times 37{\cdot}5 + 216 \times 42{\cdot}5 + 178 \times 47{\cdot}5 + 102 \times 52{\cdot}5 + 12 \times 56\} \\ &\div \{41 + 183 + 233 + 245 + 244 + 235 + 216 + 178 + 102 + 12\}\end{aligned}$$

$$= \tfrac{56140}{1689} = 33 \text{ small divisions} = 3{\cdot}3 \text{ inches.}$$

$$\begin{aligned}\bar{x} = \{&41 \times 22{\cdot}5 + 183 \times 29 + 233 \times 31 + 245 \times 32{\cdot}5 + 244 \times 33{\cdot}5 + 235 \times 34 \\ &+ 216 \times 35 + 178 \times 35{\cdot}5 + 102 \times 36 + 12 \times 36\} \div 1689\end{aligned}$$

$$= \tfrac{55562}{1689} = 33 \text{ small divisions} = 3{\cdot}3 \text{ inches.}$$

Application of the Calculus to Centres of Gravity.

Finally the last set of formulae become in the limit when we are seeking to estimate the accurate position of the Centre of Gravity in the case of a continuous body :

$$\bar{x}=\frac{\int x\,dm}{\int dm},$$

$$\bar{y}=\frac{\int y\,dm}{\int dm},$$

where $\int dm$ = the total mass of the body.

Example 1. *To find the position of the centre of gravity of an isosceles triangle of altitude* 10 *cm. and base* 6 *cm.*

Consider a small area bounded by the two lines PQ and P′Q′ each parallel to the base AB and very close to one another. Then the area of PQQ′P′ is PQ × NN′ very approximately.

Let $HN = x$ and $NN' = \delta x$.

Then, since $\frac{PQ}{AB}=\frac{CN}{CH}$,

$$\therefore\ \frac{PQ}{6}=\frac{10-x}{10};$$

$$\therefore\ PQ=\tfrac{3}{5}(10-x),$$

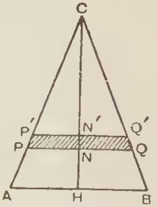


FIG. 85.

and area PQQ′P′ $=\frac{3}{5}(10-x)\,\delta x$ sq. cm. very approximately.

Hence, applying the formula and noticing that the C.G. must lie on CH by reason of symmetry, we obtain

$$GH=\bar{x}=\frac{\int_0^{10} x\times\frac{3}{5}(10-x)\,dx}{\int_0^{10}\frac{3}{5}(10-x)\,dx}=3\tfrac{1}{3}\text{ cm.},$$

where G is the C.G.

Example 2. *To find the* C.G. *of a solid hemisphere of radius a.*

Let PQ and P′Q′ represent two sections of the hemisphere very close together and each parallel to the central section HK. Let AO be the axis of symmetry.

Let the perpendicular distance of the section PQ from the centre be **x**.

Then the area of the section PQ is $\pi(a^2 - x^2)$, and the volume intercepted between the two sections PQ and P′Q′ is very approximately $\pi(a^2 - x^2)\,\delta x$ where δx is the distance apart of the two sections.

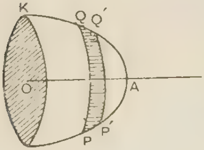


FIG. 86.

Now, from symmetry, the C.G. evidently lies on the axis of symmetry OA, and if $\bar{x}$ be the distance from O of the C.G., we have from the formula

$$\bar{x} = \frac{\int_0^a x \times \pi(a^2 - x^2)\,dx}{\int_0^a \pi(a^2 - x^2)\,dx} = \frac{3a}{8}.$$

2. Guldinus and Pappus' Theorem.

If a plane area revolve about an axis lying in its plane but not intersecting it, the volume contained by the surface described is equal to the generating area multiplied by the distance traversed by the centre of gravity of the area during the revolution.

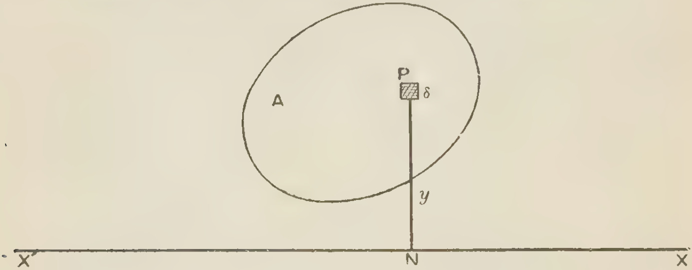


FIG. 87.

Let X′X be the axis and A the generating area. Consider a small area δA situated at P and let $PN = y$ be the perpendicular from P to X′X.

Then the volume described by δA during the revolution is $2\pi y \times \delta A$.

Hence the total volume $= \int 2\pi y\, dA = 2\pi \int y\, dA = 2\pi y A$,

where y is the perpendicular distance of the centre of gravity of A from X′X.

Example. *To find the volume of a "tore" or "anchor-ring" (i.e. the body generated by a circle revolving about a line lying in its plane).*

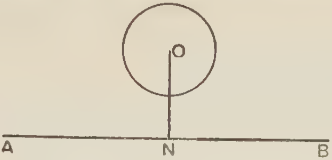


FIG. 88.

Let the radius of the circle be **a**, and let the perpendicular distance of the centre O from the line about which the circle revolves be **c**.

Then the area of the circle is πa^2 and the distance traversed by the C.G. of the generating circle during the revolution is $2\pi c$.

Hence the volume of the anchor-ring $= \pi a^2 \times 2\pi c = 2\pi^2 a^2 c$.

[*N.B.*—A child's teething-ring is a familiar example of a tore.]

PART II. MOMENTS OF INERTIA.

3. The following definition of the "moment of inertia" of a system of particles of masses $m_1, m_2, m_3, \ldots, m_n$ about an assigned straight line is given in treatises on Dynamics: **The mass of each particle is multiplied by the square of the perpendicular distance of the particle from the given straight line, and the sum of the products thus got is taken. The result obtained in this way is called the "Moment of Inertia" of the system of particles about the given straight line.**

If I denote the Moment of Inertia of the system of particles about the given straight line we shall have,

$$I \equiv m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots + m_n r_n^2,$$

where "r" is the perpendicular distance of the particle "m" from the given straight line.

If the system of particles be replaced by a single particle whose mass is their combined mass, and if this single particle be placed at a distance k from the assigned straight line so that the moment of inertia of the single particle about the given line is equal to the moment of inertia of the original system of particles about that line, k is called the "Radius of Gyration" of the assigned system of particles.

We shall then have,

$$(m_1+m_2+\ldots+m_n)k^2 \equiv \mathrm{I} \equiv (m_1r_1{}^2+m_2r_2{}^2+\ldots+m_nr_n{}^2).$$

If the system of particles be rotating about the given straight line with an angular velocity ω, the Kinetic Energy of the system is equal to $\frac{1}{2}\mathrm{I}\omega^2$.

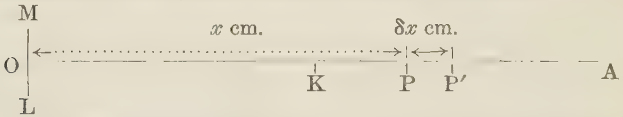
If we are dealing with the case of a continuous body instead of a system of separate particles, the above formulae become :

$$\mathrm{I}=\int r^2\,dm. \qquad k^2=\frac{\int r^2\,dm}{\int dm}.$$

Example 1. *Find the Moment of Inertia of masses* 5, 7, 8, 3, 9 *grm. distant* 2, 4, 5, 6, 9 *cm. from the axis of rotation.*

$$\begin{aligned}\mathrm{I}&=5.2^2+7.4^2+8.5^2+3.6^2+9.9^2 \text{ C.G.S. units}\\&=20+112+200+108+729 \text{ C.G.S. units}\\&=1169 \text{ C.G.S. units.}\end{aligned}$$

Example 2. *To find the moment of inertia of a thin bar of iron of mass* M *grm. and length l cm., about an axis of rotation* LM *perpendicular to the bar and passing through an end* O *of the bar.*



Let OA be the bar, whose length is l cm. and whose mass is M grm., O being the end through which passes the axis of rotation.

Let OP $=x$ cm.; PP′ $=\delta x$ cm.

Then the mass of PP′ is $\frac{\mathrm{M}}{l}\,\delta x$ gram.

$\therefore$ M.I. of PP′ about LM $=\frac{\mathrm{M}}{l}\delta x\times x^2$ (approx.) ;

$\therefore$ M.I. of the whole bar $=\Sigma\frac{\mathrm{M}}{l}\,\delta x\times x^2$ (approx.) ;

or, accurately, M.I. of the whole bar about LM

$$=\int_0^l \frac{\mathrm{M}}{l}x^2dx=\frac{\mathrm{M}}{l}\frac{l^3}{3}=\mathrm{M}\frac{l^2}{3} \text{ C.G.S. units.}$$

Corollary. *The* M.I. *of the rod about a line through its centre perpendicular to its length is* $\mathrm{M}\frac{l^2}{12}$ C.G.S. *units.*

Example 3. *Calculate approximately by graphical methods the moment of inertia of the subjoined figure about the axis* X′X.

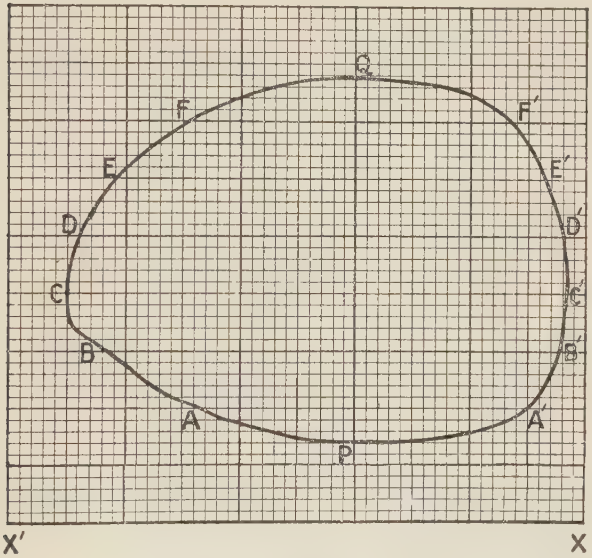


FIG. 89.

Name of section	APA′	ABB′A′	BCC′B′	CDD′C′
Area of section	59	175	212	214
Distance of centre of gravity of section from X′X	9	13	17·5	22·5

Name of section	DEE′D′	EFF′E′	FQF′
Area of section	199	164	73
Distance of centre of gravity of section from X′X	27·5	33	36·5

Moment of Inertia

$$=59\times(9)^2+175\times(13)^2+212\times(17\cdot5)^2+214\times(22\cdot5)^2$$
$$+199\times(27\cdot5)^2+164\times(33)^2+73(36\cdot5)^2$$
$$=633960 \text{ units.}$$

Example 4. *To find the* M.I. *of a rectangular lamina about one of its sides.*

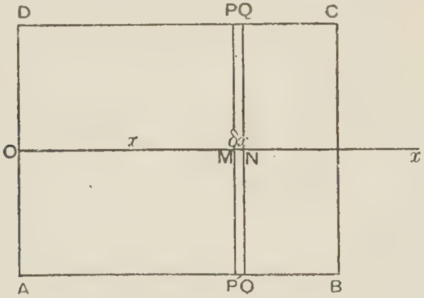


FIG. 90.

Let ABCD be the rectangle AB $=a$ cm., AD $=b$ cm. and the mass of the lamina M grams.

Suppose we require M.I. about AD.

Let Ox bisect AD at right angles.

Let OM $=x$ cm., MN $=\delta x$ cm., and let PP′, QQ′ pass through M and N respectively and be parallel to AD.

Then the mass of PP′Q′Q is $b\,\delta x\times\dfrac{M}{ab}$ grams.

$\therefore$ M.I. of PP′Q′Q about the axis AD is $\simeq\dfrac{M}{ab}\,b\,\delta x\times x^2$;

$\therefore$ M.I. of whole rectangular lamina about AD $\simeq\Sigma\dfrac{M}{a}\,\delta x\times x^2$;

or, accurately, the M.I. of the lamina about AD $=\displaystyle\int_0^a\frac{M}{a}x^2dx=M\frac{a^2}{3}$.

Example 5. *To find the Radius of Gyration k of a circular disc of radius a cm. whose mass is* M *grams about an axis of rotation passing through the centre of the disc and perpendicular to its plane.*

Let OA $=a$ cm., OP $=r$ cm., PQ $=\delta r$ cm. Then the area of the annulus contained between the circles whose radii are OP and OQ respectively is $2\pi r\,\delta r$ (approx.) sq. cm., and its mass is

$$\frac{M}{\pi a^2}\cdot 2\pi r\,\delta r$$

$$=\frac{2M}{a^2}r\,\delta r \text{ gram. (approx.).}$$

Hence the M.I. of the circular disc about an axis through O perpendicular to its plane is

$$\int_0^a r^2\times\frac{2M}{a^2}r\,dr=\frac{Ma^2}{2}\text{ C.G.S. units.}$$

Hence $k^2=\frac{a^2}{2}$, *i.e.* $k=\frac{a}{\sqrt{2}}$. The radius of gyration is therefore $\frac{a}{\sqrt{2}}$.

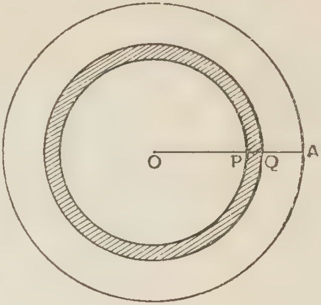


FIG. 91.

Example 6. *To find the Radius of Gyration k of a sphere, radius a cm. and mass* M *grams, about one of its diameters as axis of rotation.*

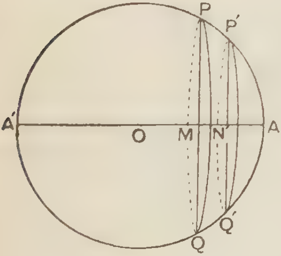


FIG. 92.

Draw two planes PQ and P′Q′ perpendicular to the diameter AOA′ cutting it at M and N. Let

OM $=x$ cm., MN $=\delta x$ cm., PM $=y$ cm.

Then the volume of the slice PQQ′P′

$=\pi y^2\,\delta x$ c.c. very approx.

$\therefore$ the mass of the slice PQQ′P′

$$=\frac{M}{\frac{4}{3}\pi a^3}\cdot\pi y^2\,\delta x\text{ grams very approx.}$$

$\therefore$ M.I. of the slice PQQ′P′ about AOA′, by Example 5 above,

$$=\frac{M}{\frac{4}{3}\pi a^3}\cdot\pi y^2\,\delta x\cdot\frac{y^2}{2},\text{ very approx., C.G.S. units.}$$

$\therefore$ M.I. of the sphere about AOA′

$$=\int_{-a}^{a}\frac{3M}{8a^3}y^4\,dx=\int_{-a}^{a}\frac{3M}{8a^3}(a^2-x^2)^2\,dx\quad(\because\ x^2+y^2=a^2)$$

$$=M\cdot\tfrac{2}{5}a^2.$$

$$\therefore\ k=a\sqrt{\tfrac{2}{5}}\text{ cms.}$$

4. Rapid methods of calculating moments of inertia.

In the practical cases that present themselves, it is usually easy to calculate by the above methods the moment of inertia and radius of gyration, if the axis of rotation passes through the centre of gravity of the body. If, however, the axis of rotation does not pass through the centre of gravity, the following theorem enables us to calculate the required moment of inertia easily and quickly :

The moment of inertia of a body about any axis is equal to the moment of inertia of the body about a parallel axis through the centre of gravity plus the moment of inertia of the body supposed concentrated at its centre of gravity about the original axis.

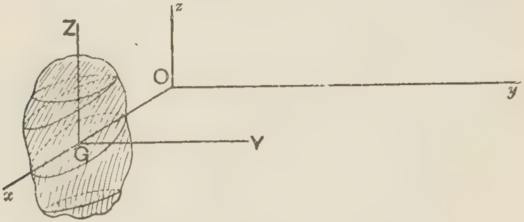


FIG. 93.

Let the given axis be chosen as the axis of z, and let the plane xOy be drawn passing through the centre of gravity and perpendicular to Oz. Let the axis of x be chosen to pass through the centre of gravity G.

Let $OG = c$.

Let axes GY and GZ parallel respectively to Oy and Oz be taken.

Then the moment of inertia of the body about Oz is $\int (x^2+y^2)\,dm$ ($\because x^2+y^2$ is equal to the square on the perpendicular distance from dm to Oz if dm be supposed placed at the point (x, y)). ....(1)

Now, if we refer to GY and GZ as axes, we obtain

$$\left.\begin{array}{l} x = X + c, \\ y = Y, \\ z = Z. \end{array}\right\} \quad \text{..........(2)}$$

Hence, by (1) and (2), the required moment of inertia becomes

$$\int\{(X+c)^2+Y^2\}\,dm=c^2\int dm+2c\int X\,dm+\int(X^2+Y^2)\,dm \quad \text{....(3)}$$

Now $c^2\int dm$ is the moment of inertia about Oz of a particle whose mass is the same as that of the body concentrated at the centre of gravity G. ....(4)

$\int X\,dm=0$ because the plane $X=0$ in the second system of coordinates passes through the centre of gravity G, thus giving X equal to zero (see Chap. X. Art. 1). ....(5)

$\int(X^2+Y^2)\,dm$ is the moment of inertia of the given body about GZ.

Hence the moment of inertia of the body about Oz *is equal to the moment of inertia of the body about* GZ *plus the moment of inertia about* Oz *of the body supposed concentrated at the centre of gravity* G.

A few examples will exhibit the practical utility of this theorem. It is assumed that in the case of the chief bodies used in engineering and mechanical work, the moments of inertia about the principal axes through the centres of gravity have been calculated and tabulated once for all.

Example 1. *Given that the moment of inertia of a thin bar of iron of mass* M *grm. and length* l *cm. about an axis of rotation through the centre of gravity and perpendicular to the bar is* $\frac{Ml^2}{12}$. *To deduce the moment of inertia about a parallel axis through the end of the rod.*

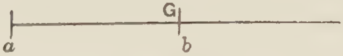


FIG. 94.

Let a, b be axes perpendicular to the rod through its end and middle respectively.

The moment of inertia of the rod about $b=M\frac{l^2}{12}$.

The moment of inertia of M situated at G about $a=M\frac{l^2}{4}$.

Hence, adding, the moment of inertia of the rod about $a=M\frac{l^2}{3}$.

Example 2. *Given that the moment of inertia of a circular lamina about a diameter is* $\frac{Ma^2}{4}$, *to find the moment of inertia about a tangent to its circumference.*

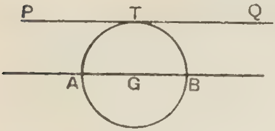


FIG. 95.

Let PQ be an axis touching the circle at T.

Let AB be the diameter parallel to PQ.

Then the moment of inertia about AB is $M\frac{a^2}{4}$, and the moment of inertia of M at G about PQ is Ma^2.

Hence, adding, the moment of inertia of the lamina about PQ is $M.\frac{5a^2}{4}$.

Example 3. *A big spherical ball is hinged to a beam and hangs suspended above a shop door. Find its radius of gyration as it swings about in the wind.*

The moment of inertia of the ball about a diameter is $M.\frac{2a^2}{5}$.

The moment of inertia of M concentrated at the centre about the hinge is Ma^2.

Hence, adding, the moment of inertia of the ball about the hinge is $M.\frac{7a^2}{5}$.

Hence the radius of gyration of the ball about the hinge is $a\sqrt{\frac{7}{5}}$.

5. A useful theorem on the moments of inertia of a lamina.

Let two perpendicular axes lying in the lamina intersect at O.

Then the moment of inertia of the lamina about $x'x$

$$=\int y^2\,dm. \qquad\qquad (1)$$

Also the moment of inertia of the lamina about $y'y$

$$=\int x^2\,dm. \qquad\qquad (2)$$

Finally the moment of inertia of the lamina about an axis through O perpendicular to its plane is

$$\int (x^2+y^2)\,dm. \qquad\qquad (3)$$

But $$\int (x^2+y^2)\,dm = \int x^2\,dm + \int y^2\,dm.$$

Hence the moment of inertia of a lamina about an axis perpendicular to its plane is equal to the sum of the moments of inertia about two perpendicular axes lying in the lamina and such that the original axis passes through their point of intersection.

Example 1. *To find the* M.I. *of a circular hoop about its diameter.*

Let I denote the required moment of inertia of a circular hoop of radius a and mass M about its diameter. Plainly the M.I. of the hoop about an axis through its centre at right angles to its plane is Ma^2. Hence, by the foregoing theorem, if we take two diameters perpendicular to one another, we obtain $I+I=Ma^2$, *i.e* $I=M\frac{a^2}{2}$.

Example 2. *A rectangular lamina* ABCD, *whose sides are* 7 *cm. and* 4 *cm. respectively, hangs in a vertical plane and swings freely about a horizontal pivot driven through* A. *Calculate the moment of inertia about the pivot if the mass of the lamina be* 120 *grams.*

We have:

M.I. of the lamina about AB is $\frac{Mb^2}{3}=120\times\frac{4^2}{3}$ C.G.S. units.

M.I. of the lamina about AD is $\frac{Ma^2}{3}=120\times\frac{7^2}{3}$ C.G.S. units.

Hence the moment of inertia about the pivot which is perpendicular to its plane $=\frac{120}{3}(4^2+7^2)=40\times 65=2600$ C.G.S units.

PART III. CENTRES OF PRESSURE.

6. It is proved in treatises on Hydrostatics that non-viscous fluids exert on flat surfaces with which they are in contact an intensity of pressure which is at every point perpendicular to that surface. We may find the position of the "Centre of Pressure" (ξ, η) by taking two convenient rectangular axes lying in the flat surface and using the formula

$$\xi=\frac{\int px\,dA}{\int p\,dA}, \qquad \eta=\frac{\int py\,dA}{\int p\,dA},$$

where p is the intensity of the fluid-pressure at any particular point of the flat surface and δA is the element of its Area.

Example 1. *A rectangular box, whose lid is* 10 *cm. by* 8 *cm., is filled with a gas whose pressure throughout is* 1000 *grams wt. per sq. cm. Calculate the turning-moment due to gas-pressure on the hinges of the lid.*

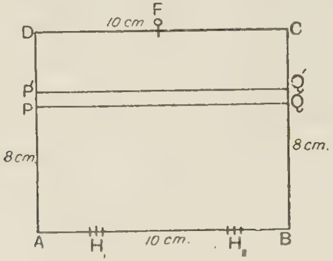


FIG. 96.

Let the above diagram ABCD represent the lid of the box, the hinges being at H_1 and H_2 respectively and the fastener at F. Let PQQ′P′ be a small rectangular area of width δx cm., whose sides are parallel to the sides of the box. Let $AP = BQ = x$ cm.

Then the pressure on PQQ′P′ is $1000 \times 10 \times \delta x$ grm. wt. and the "twisting-effect" or "moment" about the line of hinges AB is measured by $1000 \times 10 \times \delta x \times x$ grm.-cm.

Thus the total turning-effect of the pressure of the gas on the lid about AB is (very approximately)

$$\sum_0^8 1000 \times 10 \times \delta x \times x,$$

$$\textit{i.e.} \text{ accurately } \int_0^8 1000 \times 10 \times dx \times x \text{ grm.-cm.},$$

$$\textit{i.e.} \quad 10{,}000 \int_0^8 x\,dx \text{ grm.-cm.},$$

$$\textit{i.e.} \quad 320{,}000 \text{ grm.-cm.}$$

Corollary. The total pressure of the gas on the lid

$$= 1000 \times 10 \times 8 \text{ grm. wt.} = 80{,}000 \text{ grm. wt.}$$

Hence the distance of the C.P. from the line of the hinges

$$= \frac{320{,}000}{80{,}000} \text{ cm.} = 4 \text{ cm. (which is otherwise evident).}$$

Example 2. *Find the position of the centre of pressure of a triangular plate* ABC *subjected to a gaseous pressure of* p *lb. wt. per sq. in. on one side and a vacuum on the other, given that* CA = CB = 5 *ft. and* AB = 6 *ft.*

Let BO = OA, then CO is perpendicular to BA, and by symmetry it is clear that the centre of pressure lies on CO.

Divide the triangle into strips parallel to AB, and consider that one of width δx ft. which is distant x ft. from AB. Let ξ ft. be the distance of the centre of pressure from AB.

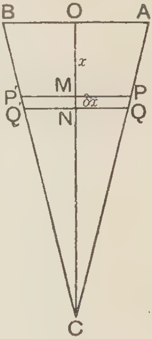


FIG. 97.

Then, by the formulae for calculating the centre of pressure,

$$\xi = \frac{\int_0^4 x \times p \times \mathrm{P'P}\, dx}{\int_0^4 p \times \mathrm{P'P}\, dx} = \frac{\int_0^4 x(4-x)\, dx}{\int_0^4 (4-x)\, dx}$$

(cancelling the constant pressure **p**)

$$= 1\tfrac{1}{3} \text{ ft.}$$

Example 3. ABCD *represents a "hatch" to run off the surplus water from a mill-dam. It is* 4 *ft. wide and* 6 *ft. deep. To find the position of the centre of pressure, given that the water is level with* AB.

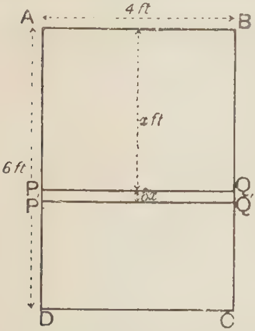


FIG. 98.

Then the intensity of pressure at any point x feet deep on the inside of the hatch is due to the effect of the atmospheric pressure plus the pressure due to the depth of the water. Also, the intensity of the pressure at any point on the outside of the hatch is due to the atmosphere only. Hence the effect of the atmospheric pressure on the hatch from within cancels that from without. Hence the *effective* intensity of pressure p lb. wt. per sq. ft. at a point x feet deep is due to the depth of the water only, *i.e.* $p = 62{\cdot}3x$ lb. wt. per sq. ft. (since a cubic foot of water weighs 62·3 lb. very approximately).

Hence, applying the formula,

$$\xi = \frac{\int_0^6 xp \times 4dx}{\int_0^6 p \times 4dx} = \frac{\int_0^6 62{\cdot}3x \times x\, dx}{\int_0^6 62{\cdot}3x\, dx} = \frac{\int_0^6 x^2\, dx}{\int_0^6 x\, dx} = 4 \text{ ft.}$$

Example 4. *To find the centre of pressure on the vertical triangular ends of an open vessel full of water.*

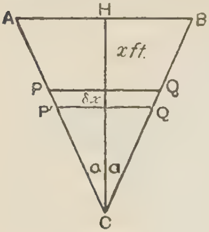


FIG. 99.

As before, the atmospheric pressures from within and without cancel.

Also the element of area to be dealt with is plainly PQ δx.

$$\text{Hence } \xi = \frac{\int_0^h xp \times \text{PQ}\, dx}{\int_0^h p \times \text{PQ}\, dx} = \frac{\int_0^h x^2(h-x)\, dx}{\int_0^h x(h-x)\, dx}$$

(where HC $= h$)

$= \frac{h}{2}$, *i.e.* the centre of pressure is half-way down the median.

EXAMPLES X.

CENTRES OF GRAVITY.

1. Find the C.G. of a triangular lamina whose sides are 7 cm., 7 cm., 4 cm.

2. Find the C.G. of a lamina in the form of a trapezium whose parallel sides are 9 cm. and 15 cm. respectively and at a distance 8 cm. apart, the whole figure being symmetrical about the line joining the mid-points of the parallel sides.

3. ABCD is a quadrilateral in which AB is parallel to DC and AD perpendicular to AB. Find the position of its C.G. given that AD $=4$ cm., DC $=7$ cm., AB $=10$ cm.

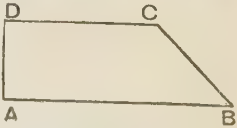


FIG. 100.

4. Find the C.G. of the portion of the area of the parabola $y^2=40x$ cut off by the line $x=10$.

5. Find the C.G. of the area enclosed by the curve $y^2=8x^3$ and the line $x=2$.

6. Find the C.G. of the area enclosed by $x^2=7y^3$ and the straight line $y=7$.

7. Find the C.G. of a pyramid standing on a square base whose side is 5 inches, the perpendicular from the apex of the pyramid on the base, passing through the intersection of the diagonals of the square and being 8 in. long.

8. A pyramid stands on a rectangular base 12 in. by 5 in. and its apex is on the perpendicular to the base through the intersection of the diagonals of the base, and is 20 in. above the base. Find its C.G.

9. Find the C.G. of a regular tetrahedron each of whose edges is 9 inches long.

10. Find the C.G. of a hollow cone h feet high and semivertical angle α.

11. Find the C.G. of the canvas of a tent which is of the shape of a cone 7 ft. high and semivertical angle of 45° standing on a cylinder $1\frac{1}{2}$ ft. high and cross-section a circle of 7 ft. radius.

12. Find the C.G. of the smaller portion of a solid sphere of radius 10 cm. cut off by a plane distant 4 cm. from its centre.

13. Find the C.G. of the smaller portion of a solid sphere of radius a cm. cut off by a plane distant b cm. from its centre.

14. Find the C.G. of the solid figure formed by the revolution of $y^2=4ax$ about the axis of x and by the revolution of $x=b$.

15. Find the C.G. of the solid figure formed by the revolution about the axis of x of that half of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which lies to the right of the axis of y.

16. Find the C.G. of the solid figure formed by the revolution about the axis of y of that half of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which lies above the axis of x.

17. A round bar of iron 6 ft. long gradually tapers from a cross-section of $\frac{1}{2}$ inch to a point. Find the position of its C.G.

18. An ordinary bucket is 12 inches across the bottom and 16 inches across the top, while the slant side is 14 inches. Find the C.G. of the water contained in the bucket when full.

19. If the thickness of the material used in the bucket is $\frac{1}{5}$ inch, find the C.G. of the empty bucket.

MOMENTS OF INERTIA.

20. A piece of wire whose mass is 0·42 gram per cm. length is bent into the form of a rectangle 30 cm. long and 20 cm. wide. Find the M. of I. of the wire rectangle (i) about the shorter side, (ii) about the longer side.

21. A bar of iron whose cross-section is negligible in comparison with its length is 4 ft. long and weighs 10 lb. It rotates about a vertical axis passing through one end, to which it is inclined at 40°. Find the M.I. of the iron bar about the axis of rotation.

22. A piece of wire whose mass is 0·4 gram per cm. length is bent into the shape of an equilateral triangle whose side is 10 cm. Find the M.I. of the wire triangle about one of its sides.

23. Find the M.I. of a trap-door 2 feet square and weighing 15 lb. (about the line of its hinges).

24. The lid of a desk is 2 ft. wide, 1 ft. 6 in. deep, and weighs 7 lb. Find its moment of inertia (about the line of its hinges).

25. ABCD and EFGH are the two wooden arms of a figure used for scaring birds from seeds. The wind causes them to revolve about ADEH as axis of rotation. If each arm weighs 6 oz. and measures 8 in. by 2 in., find the M. of I. of the arms.

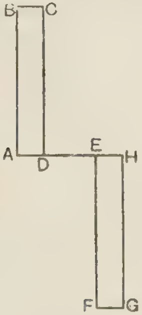


FIG. 101.

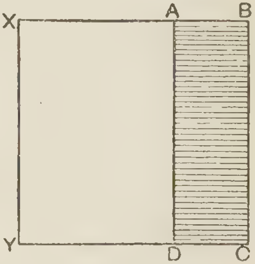


FIG. 102.

26 ABCD represents the solid part of the paddle of a paddle-wheel of a boat. It is connected by solid iron rods AX, DY with XY an iron axle, about the axis of which the paddles rotate.

AB = 2 ft., AX = 4 ft., AD = BC = 6 ft. ABCD weighs 35 lb., AX and YD each weighs 5 lb. and the axle XY 50 lb.

If there are ten paddles on the wheel, find (1) the M. of I. of the wheel, (2) its radius of gyration, (3) the kinetic energy of the wheel when making 2 revolutions per minute.

27. A wheel is formed of wire weighing 2 grams per cm. length for its rim and 0·4 gram per cm. length for each of its 12 spokes. If the radius of the wheel is 9 cm., find (1) its M. of I. about its axis of rotation, (2) its radius of gyration, (3) its kinetic energy when the wheel is making one revolution per sec.

28. In a laboratory experiment, to determine the acceleration due to gravity, a circular brass disc 10 cm. in diameter and 2 mm. thick has an axis soldered to its centre at right angles to its plane, by which it rolls down two metre scales inclined to the horizontal. If the brass weigh 8·1 grams per cu. cm., find (1) the M.I. of the disc about its axis of rotation, (2) the radius of gyration of the disc, (3) the kinetic energy of the disc when making n revolutions per sec.

29. A circular disc of lead is 8 cm. in diameter and 1 cm. thick, and it rolls down an inclined plane. Lead weighs 11·35 grams per cu. cm. Find the M.I. of the disc (1) about an axis through its centre perpendicular to its circular face, (2) about the parallel to this line on the inclined plane, about which it is instantaneously rotating at any moment.

30. A circular furnace door of iron has a diameter of 15 inches and weighs 50 lb. It is hinged in such a way that it turns about an axis which is a tangent to the circle. Find (1) its M.I. about this axis, (2) its radius of gyration, (3) its kinetic energy when moving at the rate of 50° per sec.

31. Find the M. of I. of a rectangular magnet 10 cm. long, 1·5 cm. wide, 0·5 cm. thick about a vertical axis through its C.G., one of its largest rectangular faces being horizontal. The mass of the magnet is 56 grams.

32. In an experiment carried out with the magnet in Question 31, a rectangular piece of brass is used 6 cm. long, 1 cm. wide, 0·4 cm. thick and weighing 19·36 grams. One of the largest faces of the brass is laid symmetrically upon the magnet so that the C. of G. of the brass lies vertically above the C.G. of the magnet. What is the M.I. of the system about the same axis as in Question 31 ?

33. Find the M.I. of a solid circular cylinder, length $2l$ cm. and a cm. radius of cross-section, whose mass is M grams about its axis of figure.

34. Find the M.I. of the cylinder in Question 33 about any diameter of the circle at one of its ends.

35. Find the M.I. of a magnetised knitting needle 24 cm. long, whose cross-section is a circle with radius 1 mm., about an axis perpendicular to the needle through its C. of G., the mass of the needle being 5·76 grams.

36. Find the M.I. of a circular cylindrical magnet 10 cm. long and diameter of cross-section 1 cm., weighing 236 grams, about an axis through its C. of G. and perpendicular to its axis of figure.

37. Find the M.I. of a hollow cylindrical magnet 14 cm. long, the external and internal diameters of cross-section being 1·2 cm. and 1 cm. respectively and its mass 38·72 grams, about an axis through its C.G. and perpendicular to its axis of figure.

38. What is the M.I. of a cone of height h cm. and semivertical angle α and mass M grams about its axis of figure ?

39. A frustum of a cone is contained between two circles of radii a and b cm. ($a > b$), and the distance between the circular ends is k cm. The mass of the frustum is M grams. Find the M. of I. about the axis of figure.

40. A sphere 10 cm. diameter and weighing 542 grams is attached to the end of a fine wire 1 metre long and suspended to form a pendulum. What is (1) M.I. of the sphere about the axis of rotation; (2) the radius of gyration?

41. A sphere 1 ft. in diameter and mass 2 lb. rolls down an inclined plane. What is the M.I. of the sphere about the straight line round which it is instantaneously rotating at any moment?

42. The flywheel of a Brennan gyroscope is a steel cylinder 6 inches in diameter and weighing 10 lb. When running at full speed it makes 100 revolutions per sec. Find (1) its radius of gyration, (2) its kinetic energy at full speed.

43. A steel ring or annulus of rectangular cross-section is formed by cutting out from the centre ot a disc of thickness d cm. and radius r_1 cm. a disc of the same thickness and radius r_2 cm. (see figure). If M grams be the mass of the ring, prove that the M. of I. of the disc about an axis through its centre and perpendicular to its plane is $\frac{M}{2}(r_1^2+r_2^2)$ C.G.S. units.

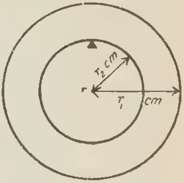


FIG. 103.

44. The simplest rigid pendulum for accurate experiment is now considered to be a ring or annulus such as described in the preceding question (provided that $r_1^2=3r_2^2$), the ring resting anywhere on its inner circumference on a knife edge.

Find the radius of gyration of such a pendulum, the external radius being 20 cm.

CENTRES OF PRESSURE.

45. A rectangular area 5 ft. wide is immersed vertically in a liquid, the upper edge being parallel to the surface and at a depth of 11 ft., while the lower edge is at a depth of 20 ft. Disregarding atmospheric pressure, what is the depth of the centre of pressure?

46. The same rectangular area as in Question 45 is immersed in a liquid with its upper edge in the surface and its plane making an angle of 60° with the vertical. Find the depth of the centre of pressure, due to the effect of the liquid only.

47. A rectangular area 8 ft. wide is immersed so that its upper edge is parallel to the surface and at a depth of 3 feet, while its lower edge is at a depth of 10 feet and its plane makes an angle of 60° with the vertical. What is the depth of its centre of liquid pressure?

48. A triangle, with base 20 inches and each of its other sides 30 inches, is immersed vertically in a liquid, its vertex being just in the surface. Find the depth of its centre of liquid pressure.

49. The triangle of the preceding question is immersed vertically, but with its base just in the surface and vertex downwards. Find the depth of its centre of liquid pressure.

50. A triangle, base 40 inches and each of its other sides 50 inches, is immersed in a liquid with its vertex in the surface, its base parallel to the surface, and its plane making angles of (i) 32°, (ii) 47°, (iii) 59°, (iv) 64°, (v) 87° with the vertical. Find the vertical depths of the centres of liquid pressure in each case.

51. A triangle, base 1 metre and each of its other sides 1·3 metres, is immersed with its base in the surface of a liquid and its vertex downwards, its plane making angles of (i) 28°, (ii) 34°, (iii) 45°, (iv) 53°, (v) 66°, (vi) 72°, (vii) 84° with the vertical. Find the vertical depths of the centres of liquid pressure in each case.

52. A heavy iron door is in the shape of a square with a 3-foot side. It is hinged on its upper edge, and hanging vertically closes the outfall of a small canal into the sea, its lower edge being level with the bottom of the canal. It is kept closed by a horizontal force of half a ton weight applied to its bottom edge. What is the depth of the water in the canal when it is just on the point of opening, and what is the position of the centre of pressure on the door ?

53. What must be the horizontal force applied $1\frac{1}{2}$ ft. from the hinges of a rectangular iron door which just closes the outlet of a drain into the sea when the depth of water in the drain is 7 ft. ? The horizontal side of the door is 4 ft. and the vertical 3 ft. Find also the centre of pressure on the door.

54. A lock gate is 9 ft. wide and the depth of water on one side of it is 15 ft. and on the other side 8 ft. Find the resultant thrust on the gate and the point where its line of action meets the gate.

[*N.B.*—The effect of the atmospheric pressure must be taken into account in this question.]

MISCELLANEOUS EXAMPLES I.

CHAPS. I.–V.

1. If y is known to be a quadratic function of x, and we have in addition the following scheme of values:

y	1	6	35
x	1	2·5	4

express y in terms of x, and find the corresponding values of $\frac{dy}{dx}$.

(Army Adapted.)

2. A man 5 ft. 8 in. in height walks towards an electric arc street light at the rate of 4 miles per hour. The light is 25 ft. above the level of the street. Find the rate in ft. per sec. at which his shadow diminishes in length. (Army.)

3. By drawing the graph of the curve $5y = x^3 - x$, estimate roughly the position of points on it at which the tangent makes an angle of 45° with the axis of x. Then, by means of the calculus, find the points accurately. (Army Adapted.)

4. Find the maximum and minimum values of $y = x + \frac{4}{x+2}$; illustrate your results by drawing a rough graph of the function.

(M.T.)

5. A window is in the form of a rectangle surmounted by a semicircle. If the perimeter is 30 ft., find the dimensions of the window so that the greatest possible amount of light may be admitted.

(Downing.)

6. A body moves in a straight line, and its distances from a given point in the line at various times are given in the following table:

Time in seconds -	0	1	2	3	4	5	6	7	8
Distance in feet -	0	3	12	27	48	75	108	147	192

Draw a graph showing the connection between distance and time.

Derive a velocity-time curve, and find the velocity at the end of 3, 4, $7\frac{1}{2}$ seconds.

What do you infer as to the acceleration of the body?

(Army Adapted.)

7. Draw, in the same figure, the graphs of the equations

$$(1)\ v = 8 + 32t,$$

$$(2)\ s = 8t + 16t^2,$$

measuring t along a horizontal and v and s along a vertical axis.

Find two points on the axis of t, unit distance apart; draw ordinates to the first locus, and calculate the area of the trapezium so formed. Show that the measure of this area is equal to the difference between the ordinates when they are drawn from the same points to the second locus. (Army.)

8. A straight line KL is 12 cm. long. At the ends of this straight line and on the same side of it are erected perpendiculars KA 6 cm. long and LB 3 cm. long. The position of a point P in KL is specified by its distance, x, from K. Express $AP^2 + PB^2$ in terms of x, and hence determine the position of P when $AP^2 + PB^2$ is a minimum. (Army Adapted.)

9. A circular cylinder of a given capacity, closed at each end, is to be made of thin sheet metal of a uniform thickness. Find the ratio of the diameter to the length of the cylinder such that the weight of the vessel shall be a minimum. (Q.P.M.S.T.)

10. Describe, and illustrate by a figure, the noticeable features of the curve

$$y = (x - 60)(x - 61)(x - 62).$$

Find by the calculus the inclination to the axis of x of the tangent to the curve at the point where $x = 61$.

11. Equal squares are cut from the four corners of a rectangular sheet of cardboard measuring 30 inches by 14 inches, and the sides and ends are turned up so as to form an open rectangular box. Find the cubical contents of the box when the squares cut out have sides of x inches length. By aid of the calculus determine the value of x which corresponds to the largest box and the volume of this box. (Army Adapted.)

12. The amount of water in a tank at the end of t seconds from opening a tap is given in the following table:

Time t seconds -	0	5	10	15	20	30	40	50	55	60	70
Gallons of water in tank	64	56·2	49	42·3	36	25	16	9	6·25	4	1

Plot these quantities, and deduce from the curve the rate at which water was leaving the tank at the end of 15 seconds and 55 seconds. (Army.)

13. A point moves along the line Ox so that its distance from O at time t seconds is $(t^3 - 12t)$ feet. Find an expression for its velocity at this time, and sketch graphs showing the distance and velocity as functions of the time, from $t = -4$ to $t = +4$. (Army.)

14. The figure represents a triangular field, right angled at A, divided into two equal portions by a straight fence running from a point P in the side AC to a point Q in the side AB. The lengths in metres are as follows: $AB = a$, $AC = b$, $AP = x$, $AQ = z$, $PQ = y$. Express y in terms of x and z. Write down a relation between x, z, a, b, and use it to show that

$$y^2 = x^2 + \frac{a^2b^2}{4x^2}.$$

By aid of the calculus find the value of x which makes y as small as possible. (Army Adapted.)

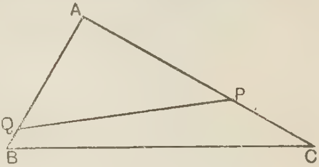


FIG. 104.

15. Draw a part of the graph of the function $y = x^3 - 2x^2 + x - 1$, plotting points for which $x = -1, 0, \frac{1}{2}, 1, \frac{3}{2}, 2$, and by aid of the calculus find accurately for what values of x the function is a maximum and a minimum. (Army Adapted.)

16. A circular cylinder with plane ends is inscribed in a sphere. Show that, when the volume of the cylinder is a maximum, it bears to the volume of the sphere the ratio of $1 : \sqrt{3}$. (Q.P.M.S.T.)

17. Find the equations to the tangent and normal to the curve $y = a + \dfrac{x^2}{4a}$ at the point for which $x = 3a$.

Find also the points in which these lines cut the axes.

(Q.P.M.S.T.)

18. In the figure, ABC represents a flower-bed in the form of an isosceles right-angled triangle whose sides AB, AC are each 12 ft. long. Parallels DE, FG are drawn at equal distances x feet from AC, AB respectively, to divide ABC into four areas, those shaded

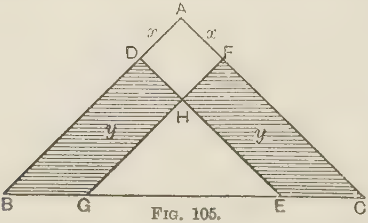


FIG. 105.

being each y square feet. Express the length of BD and GH in terms of x, and hence express y as a function of x.

By aid of the calculus find how x must be chosen so as to make y as large as possible. (Army Adapted.)

19. Find the values of A and B in the identity

$$\frac{4}{(x-4)x} \equiv \frac{A}{x-4} + \frac{B}{x}.$$

If $y = \frac{4}{(x-4)x}$, tabulate the values of y and $\frac{dy}{dx}$ for the following values of x: $-\infty$, -2, -1, 0, 1, 2, 3, 4, 5, ∞. Plot the graph $y = \frac{4}{(x-4)x}$, paying particular attention to its form when x approaches the values 0 and 4. (Army.)

20. O is a fixed point on a straight line, and P a point moving on the straight line, so that at time t seconds $OP = t^2 - 4t + 5$ centimetres. Find an expression for the ratio of the distance P moves in the interval between t and $t + \frac{1}{1000}$ to the interval $\frac{1}{1000}$. Is this expression distinguishable from the velocity at any time t for the purpose of a drawing in which 1 cm. represents a velocity of 1 cm. per sec. ?

Suppose that at every position of P a perpendicular PQ to OP is drawn representing the velocity of P on the scale mentioned above. Draw the path of Q, and mark with the numbers 0, 1, 2, 3, 4, 5 the positions of Q at times 0, 1, 2, 3, 4, 5 seconds. When is P at rest, when is it moving in the positive direction and when in the negative ? (Army.)

Repeat these calculations, using the methods of the Calculus.

21. A rectangular sheet of cardboard ABCD (see figure) is taken. Portions are cut out at the four corners, leaving only the part shaded in the figure, and this portion is folded so as to form a rectangular box with a lid. If $AB = a$, $BC = b$, and x is the length of a side of each of the two squares cut out at the two bottom corners, prove that the volume of the box is $\frac{1}{2}x(a-2x)(b-2x)$. If $a = 6$ in., $b = 12$ in., find approximately the value of x for which this volume is a maximum, and find this maximum volume in cubic inches. (Army.)

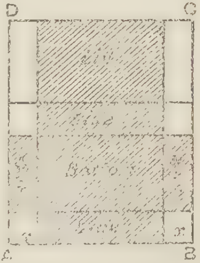


FIG. 106.

22. Find, by any method, the coordinates of the points in which the straight line $y = 2x - 8$ intersects the parabola $y^2 = 5x$.

Obtain the equations of the tangents at their points of intersection; and the coordinates of the point in which the tangents intersect.

23. The figure represents two straight roads crossing each other at right angles, A and B being places on the roads, distant a miles and b miles respectively from the crossing. Two motor cars start simultaneously from A and B towards the crossing with constant speeds u miles per hr. and v miles per hr. respectively. Show that their distance apart after t hours will be

$$d=\sqrt{\{(u^2+v^2)t^2-2(au+bv)t+(a^2+b^2)\}}\text{ miles.}$$

Taking $u=16$, $v=12$, $a=3$, $b=4$, and using the calculus, find at what time the cars are closest together and what this shortest distance is. (Army Adapted.)

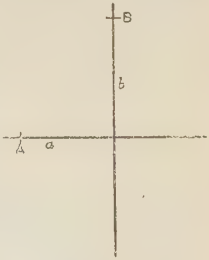


FIG. 107.

24. The following table gives observations of the Centigrade temperature of some water in a vessel. The temperature taken at intervals of 5 minutes was: 80°, 67°, 56°, 47°, 40°, 35°, 31°. Plot the temperatures with the times and join by a smooth curve.

Estimate, as nearly as you can, the rate at which the temperature was falling at the moment when the temperature was 60°, and when it was 50°, and when it was 40°.

Taking the room temperature at 20°, compare these three results, and state whether the rate of cooling was proportional to the excess of temperature above that of the room. (Army.)

25. The figure represents a rhombus made up of four rods each of length a cm. jointed together at their extremities.

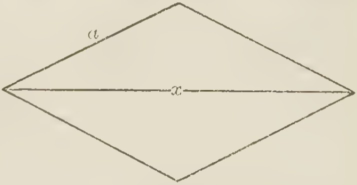


FIG. 108.

If the length of one diagonal is x cm., prove that the length of the other is $\sqrt{4a^2-x^2}$ cm., and that the area of the rhombus is $\frac{1}{2}x\sqrt{4a^2-x^2}$ sq. cm. Taking $a=8$ and using the calculus, find for what value of x the area is greatest, and calculate the length of the other diagonal in this case. From your result, what conclusion would you form as to the shape of the rhombus when its area is greatest?

(Army Adapted.)

26. A point R is taken on the side AB of a triangle ABC of area z, so that $AR = x \cdot AB$, where $x > \frac{1}{2}$. RQ is drawn parallel to BC to meet AC at Q, RH parallel to AC to meet BC at H, and QK parallel to AB to meet BC at K. Prove that the areas of ARQ and BRH are x^2z and $(1-x)^2z$ respectively (notice that they are similar to ABC), and find in similar form the area of CQK; use these results to find the area of QRHK.

By aid of the calculus find the value of x which makes the trapezium QRHK greatest. (Army Adapted.)

27. Find the value of $\frac{dy}{dx}$ at any point (x, y) of the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and write down the equation to the tangent to the curve at the point. If the tangent cut the axes Ox, Oy, in P and Q respectively, show that OP + OQ is constant and equal to a.

Take "a" equal to 4 inches, and draw a number of straight lines cutting the axes and having the sum of their intercepts equal to this. Show that the curve may then be sketched to touch these lines. (Q.P.M.S.T.)

28. Find the coordinates of the points of intersection, and the angle at which they cut, of the curves

$$2x^2 + 5y^2 = 1{\cdot}3 \quad \text{and} \quad x^2 + y^2 - 1{\cdot}3y + 0{\cdot}11 = 0. \qquad \text{(Army.)}$$

29. A measuring cylinder, open at the top, is constructed of thin metal to have a capacity of 1 litre. The radius of the circular base being r cm., the height h of the cylinder in cm., and the area S of the metal in sq. cm. are given by $h = 318{\cdot}3/r^2$, $S = 3{\cdot}142r^2 + 2000/r$. By aid of the calculus find the values of r and h which make S a minimum, and also the minimum value of S. (Army Adapted.)

30. Prove that the part of the tangent to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ intercepted between the coordinate axes is of constant length. (Downing.)

31. A straight lever, whose length is x feet and which weighs 1 lb. per foot, is held horizontally, with one end resting on a fulcrum. A weight of 32 lb. hangs from the lever at a point distant 3 inches from the fulcrum, and is supported by a vertical force P lb. at the other end of the lever.

Express P in terms of x, and by aid of the calculus find for what value of x the force P is least. (Army Adapted.)

32. Trace the curve of $y^2 = x(x-1)(2-x)$; and show that the tangents at the points on the curve given by $x = 1 + \frac{1}{\sqrt{3}}$ are parallel to the axis of x. (M.T.)

33. A ship A steams due east from a certain port P at 20 knots, setting out at the moment when another ship B, which is steaming at 15 knots towards P, is 40 nautical miles due south of P. Find a formula for the distance between the ships A and B at time t hours after A has left P. Determine the value of t for which the distance AB is least, and what this least distance is. A knot is a rate of 1 nautical mile per hour.

You may assume that when a function of t has a turning value for a certain value of t, the square of the function will also have a turning value for the same value of t. (Army.)

34. Find the slope of the tangent to the hyperbola $x^2 - y^2 = 4$ at the point $(\frac{5}{2}, \frac{3}{2})$. Hence write down the equation of the tangent at the point. If this tangent meets the lines $y = x$ and $y = -x$ at Q and R respectively, find the coordinates of Q and R and the area of the triangle QOR. (Army Adapted.)

35. Draw the curves $y = x^2$ and $y = x^3$ from $x = -1$ to $x = +1$. For each of these curves:

(1) Write down the slope of the tangent at the point whose abscissa is x, stating the numerical value at the point (1, 1), and finding the actual angle which the tangent at this point makes with the axis of x. Verify by rough measurement of the angle in the figure.

(2) Calculate the slope of the chord of each curve joining the points given by $x = 1{\cdot}0$ and $x = 1{\cdot}1$.

(3) Calculate the slope of the chord joining $x = 0{\cdot}9$ and $x = 1{\cdot}1$.

(4) Find by how much per cent. the slope of each chord calculated under (2) and (3) differs from the slope of the corresponding tangent at $x = 1$. (Army.)

36. If the sum of the lengths of the hypotenuse and another side of a right-angled triangle is given, show that the area of the triangle is a maximum when the angle between these sides is 60°. (M.T.)

37. Find the maximum and minimum values of $2x^3 - 3x^2 - 10$, and draw a graph. (Downing.)

38. Semicircles are described on AB, AC, CB. Given that the radius of AKB is r cms., and that of ALC x cm., show that the area of the unshaded portion of the figure is $\pi x(r - x)$ sq. cm.

With 9 as the value of r, calculate the area of the unshaded portion for the following values of x: 0, 1, 2, ... Tabulate your results and draw a graph to show the variation in the area, showing x full size, and representing area on the scale of 1 small division of your book for π sq. cm. What is the position of C when the area of the part

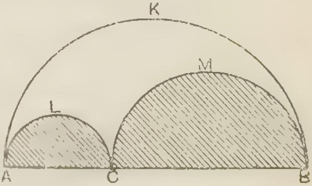


FIG. 109.

between the circles is a minimum ? a maximum ? Compare the maximum area with that of the semicircle AKB.

Regarding C as a point moving from A to B, find at what rate the area of the unshaded part is increasing when x is 3 cm., if x is increasing at the rate of 1 cm. per sec. (Army.)

39. Determine the slope of the curve $y^2(x-3)=2x$ at the point (6, 2), and find the length of the normal at that point intercepted between the curve and the axis of x. (Q.P.M.S.T.)

40. Find the cylinder of given superficial area (the ends being included) whose volume is greatest.

41. Give the equations to the tangents to the curves $x^2+y^2=4$ and $5x^2+y^2=5$ at the points where they intersect; and show that the angle at which they intersect is 37° 46′ to the nearest minute. (Q.P.M.S.T.)

42. A circular plate is expanding by heat. When the radius passes through the value 3 feet, it is increasing at the rate of 0·001 inch per sec. Find the rate of increase of the area at this moment. (Oxf. and Camb. Joint Board.)

43. A rectangular plate has its sides 10 inches and 5 inches respectively. If equal squares are cut out at the four corners, and if the sides are then turned up so as to form an open box, show that the volume of the greatest box so formed is $\frac{125}{3\sqrt{3}}$ cu. in. (Trinity.)

44. A cylinder with plane ends has a given volume. Show that its total surface is a minimum when the length of the cylinder is equal to the diameter of an end.

What are the dimensions of such a cylinder if its volume is 10 cu. ft. ? (Q.P.M.S.T.)

45. The sides of the base of a biscuit tin are in the ratio 2 : 1. Find the ratio of the height to the smaller side if the tin is to have a given content for a minimum expenditure of material. [Neglect the flanges which hold the lid on the tin.] (Oxf. and Camb. Joint Board.)

46. If s is the distance described in time t by a particle moving in a straight line, give the meanings of $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$.

If $t=as^2+2bs+c$, where a, b, c are constants and a is positive, show that the particle undergoes a retardation which at any instant is proportional to the cube of its velocity. (Oxf. and Camb. Joint Board.)

47. If a uniform beam of length $2a$ is supported at its two extremities A and B and has a weight attached to a point C in its length, where $CB = \frac{1}{2}a$, it can be shown that the bending moment at any point of the rod between A and C is measured by $(2+n)ax - x^2$, where x is the distance of the point from A and n is the ratio of the attached weight to the weight of the rod, and that at any point between C and B by $(2+3n)ax - x^2$, where x is the distance of the point from B.

Show that the bending moment at any point of the rod is greatest at C if $n > 1$, but is greatest at some point between C and the middle point of AB if $n < 1$. (Oxf. and Camb. Joint Board.)

48. Plot the curve $y = \dfrac{x^2 - x + 1}{x - 1}$ and find the maxima and minima values of y considered as a function of x.

(Oxf. and Camb. Joint Board.)

49. A man orders a plumber to make a cistern with a circular base to hold 1000 cu. ft. of water. It is to be open at the top. It has to be lined inside with sheet lead at 4½d. per sq. ft. Find the least that the cost of lining will be.

50. A closed tank to contain 120 cu. ft. of water is to be made of iron plates riveted together in the shape of a circular cylinder terminated by one hemispherical and one plane end. Find its dimensions so that the number of plates used may be a minimum.

MISCELLANEOUS EXAMPLES II.

CHAPS. VI.–X.

51. A load of 1200 lb. is hauled up a mine by a rope, and a spring balance is inserted between the load and the rope so that the lifting tension can be observed. This tension is T lb., when the distance from the bottom of the mine is x ft., and the observed values are as follows:

x	0	100	200	300	400	500	600	700	800
T	2400	2250	2030	1780	1500	1200	900	680	580

Plot a graph of force and distance, and find the whole work done in the 800 feet.

Show that up to 500 feet the load is being accelerated, and that afterwards it is slowing down. Also find its greatest velocity and its velocity when $x = 800$. (Army.)

52. Draw the two parabolas $y^2 = 4x$, $x^2 = 4y$, and find the area common to them. (Oxf. and Camb. Joint Board.)

53. If a hollow vessel of the shape formed by the revolution of $y^2 = x^3$ about the x-axis is placed under a tap from which water is flowing at 12 cu. in. per sec., at what rate is the level of the water rising in it when the depth is 1 inch, 2 inches, and 3 inches respectively?

54. Find the volume of the slice of a sphere, of radius unity, cut off by a plane at a distance c from the centre.

Calculate or estimate, as nearly as you can, the value of c for the plane which cuts off a volume equal to a quarter of the whole volume of the sphere. (Q.P.M.S.T.)

55. Prove that the area of the portion of a parabola cut off by a line at right angles to the axis, is two-thirds of the rectangle formed by this line, by two lines parallel to the axis through the ends of this line, and by the tangent at the vertex.

Find the position of the centre of gravity of the above portion of the parabola. (Q.P.M.S.T.)

56. A small tank in the shape of a triangular prism (see figure) is full of water. The end ABEF is vertical and the top ABCD is horizontal. Determine by direct integration, without using formulae, the resultant thrust on the base DCEF and the point where the resultant thrust acts. AD = BC = 96 cm., DC = AB = FE = 20 cm., AF = BE = 28 cm. (Army.)

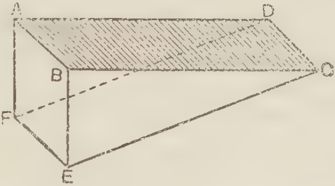


FIG. 110.

57. A uniform rod AB, length 1 metre, weight 2 kilograms, has a piece CBD of like material, length 20 cm., soldered to it at right angles at B. Find by integration the second moment (or moment of inertia) of the two rods about an axis through A, parallel to CBD. The unit in which the answer is given should be distinctly stated. (Army.)

58. Draw that portion of the graph of the equation $4y = 8x - x^4$ for which x and y are both positive: first calculating the gradient of the tangent at the points where x has the values 0, 1, 2 and the coordinates of the point where the tangent is parallel to the x-axis. Calculate the area enclosed between the curve and the x-axis. (Army Adapted.)

59. A tram car is travelling at 14 m.p.h. when the brake is applied, and the subsequent relation between the velocity and the time is given by the following table:

Time $=t$	0	1	2	3	4	5	seconds.
Velocity $=v$	14	10·54	8·37	6·9	5·82	5	m.p.h.
Time $=t$	6	7	8	9	10	seconds.	
Velocity $=v$	4·36	3·84	3·41	3·05	2·75	m.p.h.	

Plot a velocity-time curve, and deduce the distance the tram car has covered during these 10 seconds. (Q.P.M.S.T.)

60. Draw the curve $y=(1-x)(x-5)$. Find the angle which the tangent makes with the axis of x at the points where the curve cuts that axis, and verify that this angle is shown roughly correct in your figure. Find by integration the area of the portion of the curve which lies on the positive side of the axis of x.

Find also the maximum ordinate.

Find the value of $\dfrac{d^2y}{dx^2}$ at a given point of the curve. (Army.)

61. Calculate by means of the integral calculus the area above the axis of x of the curve $y=8x-x^2-7$.

62. Steam at a pressure of 120 lb. per sq. in. is admitted to a cylinder, cross-sectional area 300 sq. in., stroke 30 in. Halfway down the stroke steam is cut off so that no more is admitted to the cylinder, and for the remainder of the stroke the relation between the pressure p and the volume occupied by the steam is that $pv^{1\cdot4}$ is constant.

Find an expression for the thrust of the steam upon the piston during the first half of the stroke, and an expression for the thrust at any point of the second half.

Find the value of the second expression when the piston has travelled 20, 25 and 30 inches.

Find by integration the total work done by the steam during one stroke. (Army Adapted.)

63. Sketch the curve $a^4y^2=(a^2-x^2)^3$, and find the volume of the solid which is formed by the revolution of this curve about the axis of x. (Q.P.M.S.T.)

64. A uniform solid is bounded by the surface obtained by revolving the curve

$$y^2 = ax^2 + 2bx + c$$

about the axis of x. A slice is cut from the solid by two plane sections perpendicular to the axis, at a distance h apart. If A and B be the areas of the two plane faces of the slice, show that the distance of the centroid of V from the face A is equal to

$$\frac{h}{2} + \frac{(B - A)h^2}{12V}.$$

(M.T.)

65. A drain of rectangular section is closed by a square door which measures 2 feet in the edge, is hinged about the upper horizontal edge, and is inclined at 45° to the horizontal when closing the drain. What must be the weight of the door if it is designed to open when the water has risen halfway up it? A cu. ft. of water weighs 62·3 lb. (Army.)

66. Prove that the moment of inertia of a uniform rod about an axis intersecting the rod and at right angles to it is $M(\frac{1}{3}a^2 + b^2)$ where $2a$ is the length of the rod, M its mass and b the distance of the axis from the middle point. (Q.P.M.S.T.)

67. One end of a steel bar was held rigidly in a clamp, and a twisting moment was applied to the other end and increased until the bar broke. Readings of the twisting moment and of the angle through which the bar twisted were taken at frequent intervals as follows:

Twisting moment, lb.-inches	0	2000	3000	4000	5000	5500	6000	6200
Angle, degrees -	0	5	50	180	480	900	1600	1900

The bar broke when the twisting moment reached 6200 lb.-inches.

Plot a graph showing the relation between the twisting moment and the angle of twist.

The total work done in breaking the bar is calculated by taking the product of the mean twisting moment and the total angle of twist in radians. Obtain the necessary information from the graph, and calculate this total work. (Army.)

68. Sketch the curve $y = x(x-1)(x-2)$ for values of x between -1 and 3. Prove that $\int_0^2 x(x-1)(x-2)\,dx = 0$, and interpret this result with respect to the curve you have already drawn.

Find also the values of x which give the maximum and minimum values of y. (Army.)

69. It is found that the rate at which a man can dig a trench is constant for a quarter of an hour, and thereafter varies inversely as the square root of the number of hours he has been at work. At the end of 2 hours' work he is excavating at the rate of 22·5 cu. ft. per hour. Determine his rate of work at the end of the first quarter of an hour and at the end of each hour of a 4-hour spell of work. Find also his rate of work at the end of x hours. Plot a rough graph to show his rate of working at each instant of the spell, and by integration determine the number of cu. ft. of material he will excavate and his average rate of work. (Army Adapted.)

70. Find the position of the centre of gravity of a solid hemisphere which has a radius of 5 inches.

71. An equilateral triangle, each side $4''$ in length, is cut from sheet metal weighing 5 lb. per sq. ft. Find the moment of inertia of the triangle about one side, measured in "lb.-ft.^{2}" units. (Q.P.M.S.T.)

72. A rectangular plate used in connexion with a weir is entirely immersed, its upper and lower edges are horizontal, and are immersed at x cm. and y cm. respectively from the surface of the water. Determine an expression in terms of x and y only for the depth of the centre of pressure; and given that the length of the horizontal sides is a cm. and the length of the other 2 sides b cm., find an expression for the total thrust of the water on this plate. (Army.)

73. The specific heat of water s (in joules) at temperature $t°$ being given by the following table:

$t=$	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°
$s=$	4·219	4·195	4·181	4·174	4·173	4·174	4·178	4·184	4·190	4·197	4·205

show that, to heat 1 gram of water from 0° to 100° requires 418·5 joules, approximately.

[The specific heat of a substance at temperature $t°$ is $\frac{dQ}{dt}$, where Q is the quantity of heat required to heat 1 gram of the substance from some fixed temperature to $t°$.] (M.T.)

74. A car whose mass is 2000 lb. starts from rest; the resistance to the motion is equal to 50 lb. weight. When it has travelled S feet the force exerted by the engine is P lb. weight where

S	0	10	20	30	40	50	60	70	80	90	100
P	644	634	622	607	587	565	537	509	475	440	404

Find, approximately, the velocity after the car has travelled 100 feet, assuming that the acceleration due to gravity is 32 in ft.-sec. units. (M.T.)

75. Draw the curve $y=x(4-x^2)$, and find the area included between the x-axis and the part of the curve lying between $x=0$ and $x=2$.

76. Trace roughly the portion of the curve $y=8-x^3$ for which both x and y are positive.

Find the equation of the tangent at the point whose coordinates are a, b. Calculate to two significant figures the values of a, b for which the tangent is equally inclined to the axes, and the values of the intercepts it makes upon the axes.

Find by integration the area enclosed by the curve and the axes. (Army Adapted.)

77. Obtain the equation to a curve in which the tangent of the slope at any point varies as $\sqrt{x}$, and the slope is 45° when $x=1$; the curve is to pass through the point $(1, -2)$. (Army.)

78. The ellipse whose equation is $\frac{x^2}{25}+\frac{y^2}{9}=1$ revolves about the axis of x. Find (i) the volume and (ii) the position of the centre of gravity of that portion of the solid swept out which lies about the positive portion of the axis of x.

79. A dock-gate is 50 ft. high and 30 ft. broad, and is supported against the water pressure by horizontal forces acting at the top and bottom perpendicular to the face of the gate. Determine the magnitude of these forces when the water stands 40 ft. and 25 ft. deep on the two sides respectively. (Q.P.M.S.T.)

80. A heavy uniform square plate, whose side is one foot in length, is pivoted at one corner so that it can move in a vertical plane. The plate is displaced through an angle of 60° from its equilibrium position and then released. Find its greatest angular velocity in the subsequent motion. (Q.P.M.S.T.)

81. A curve has to be drawn through the points (0, 0), (1, 4), (2, 1), (3, 5). Assuming the equation of the curve to be of the form

$y = ax^3 + bx^2 + cx + d$, determine a, b, c, d, and find by integration the area of the curve between the extreme ordinates and the axis of x.

The area comes out to be equal to that of a rectangle on the same base and of altitude $2\frac{1}{2}$. Draw a figure showing the curve and the rectangle, and show that the result could be inferred from the symmetry of the figure. (Army.)

82. In finding the volume of earth in a cutting between two sections A and B, the following measurements were obtained:

	Distance from A.	Area of section of cutting in sq. ft.
At A -	0	2000 sq. ft.
	8 yards.	3000 ,,
	12 ,,	3420 ,,
	19 ,,	3960 ,,
	24 ,,	3900 ,,
At B -	30 ,,	3700 ,,

Find the volume of earth in the cutting and the average area of cross-section of the cutting. (Army.)

83. The curve $y^2 = x^3$ revolves about the axis of x forming a surface of revolution. Find the rate of increase of the volume cut off by a plane perpendicular to the axis of x when $x = 3$.

84. By integration find an expression for the volume of the portion of a sphere cut off by a plane, and deduce an expression in terms of the height and diameter of base of the portion.

Find the volume of a gasholder which consists of a cylinder 30 ft. in diameter and 20 ft. high surmounted by a spherical segment 10 ft. high. (Army.)

85. Observations of the speed of a car in miles per hour were taken at intervals of 12 seconds over a period of 2 minutes as follows:

Time in seconds	0	12	24	36	48	60	72	84	96	108	120
Speed, m.p.h. -	5	3	2·5	3	5·7	10·3	14	10·5	5·6	2	0

Plot the graph of speed and time, and from it estimate the whole distance run in the two minutes.

Also estimate the mean value of the acceleration during the period between 30 seconds and 40 seconds. (Army.)

86. The equation of a hyperbola is of the form $y^2 = A + Bx + Cx^2$. Determine the constants A, B, C, if it is given that the curve passes through the points (0, 0), (1, 0), (2, 2). (The x coordinate is written first.) Draw a rough sketch of the curve.

If this curve revolves about the axis of x, find the volume generated by the portion between the limits $x = 1$ and $x = 2$. (Army.)

87. The equation to the curve AB is $y = (a + bx)^{-n}$. Obtain an expression for the area ABCD, the coordinates of A being x_1y_1 and those of B being x_2y_2.

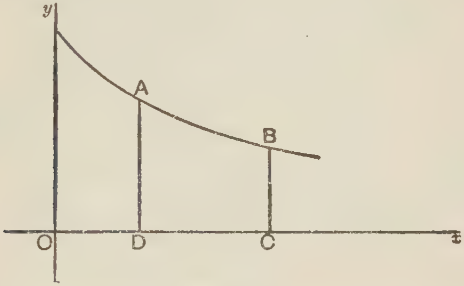


FIG. 111.

Taking $a = 0$, reduce your expression to its simplest form, and determine the number of sq. inches in the figure if $y_1 = 7{\cdot}5$ inches, $y_2 = 1{\cdot}0$ inch, $x_2 = 12$ inches, and $n = \frac{17}{16}$. (Army).

88. A uniform solid is bounded by the surface obtained by revolving the curve

$$y^2 = ax^2 + 2bx + c$$

about the axis of x. A slice is cut from the solid by two plane sections perpendicular to the axis, at a distance h apart; prove that the volume of the slice is V, where

$$V = \tfrac{1}{2}(A + B)h - \tfrac{1}{6}\pi a h^3,$$

A and B being the areas of the two plane faces of the slice. (M.T.)

89. Find for what values of x the function

$$x^3 - 9x^2 + 24x + 2$$

is (i) increasing, (ii) diminishing, respectively.

Sketch the graph of the function, and determine by integration the area bounded by the curve, the x-axis, and the ordinates at the points where the function is a maximum and a minimum. (Q.P.M.S.T.)

90. A gas is compressed in a cylinder by the movement of the piston. The relation between F the force on the piston and D the distance from the inside face of the piston to the inside end of the cylinder is given by the equation $F \, . \, D^{1 \cdot 42} = \text{constant}$. When $D = 18$ inches, F is 110 lb.

By integration find the work done (in ft.-lb.) during the compression of the gas as D decreases from 18 inches to 3 inches. (Army Adapted.)

91. How many foot-lb. of work are spent in filling a water tank of the following internal dimensions, length 20 feet, breadth 10 ft., depth 8 ft., if the bottom of the tank is 20 feet above the level of the water in the well from which the supply is drawn. (Army.)

92. A uniform rod, of length 2 ft., can turn about a horizontal axis through one end. The other end is raised to the same height as the axis and then released ; find the angular velocity when the rod has fallen to the vertical. (Q.P.M.S.T.)

93. (i) Find the centre of pressure of a rectangular plate immersed in water with one edge in the surface.

(ii) A horizontal rectangular trough 4 feet deep and 3 feet wide is closed at one end by a door hinged at the upper edge. What horizontal force is required at the lower edge of the door to prevent it from opening when the trough contains water to a depth of 18 inches ? (Q.P.M.S.T.)

94. Find the volume of the solid formed by the revolution about the axis of x of the curve $y^2 = x(a - x)$. (Q.P.M.S.T.)

95. The curve $y = 4x^{\frac{1}{2}}$ is rotated about the axis of x. Find the volume enclosed by the surface of revolution between the sections at $x = 3$ and $x = 8$.

The same curve is rotated about the axis of y. Find the volume enclosed by the surface of revolution generated by the same portion of the curve. (Q.P.M.S.T.)

96. A cone of semivertical angle 25° is hollow and is placed vertex downwards under a tap from which water is flowing into it at the rate of 12 cu. in. per sec. At what rate is the level of water rising in the cone when the depth is 20 inches ?

97. An equilateral triangle whose sides are of length a is divided into two portions of equal area by a line parallel to one side. If the whole triangle be rotated about that side, find the volumes of the solids traced out by the two portions of the triangle. (Q.P.M.S.T.)

98. The portion of the paraboloid of revolution which is formed by the revolution of $y^2 = 8x$ about the x-axis and cut off by a plane perpendicular to the axis and distant 2 cm. from the vertex, is made in iron and immersed in a vessel in the shape of a right circular cylinder of radius 6 cm. cross-section containing water to a depth of 4 cm. Through what height does the water rise in consequence of the immersion ?

99. A portion is cut off from a sphere of radius a by a plane at a distance $\frac{a}{2}$ from the centre. Show that the volume of this portion is $\frac{5}{32}$ of the volume of the sphere. (Q.P.M.S.T.)

100. The figure shows a portion of the rectangular hyperbola $x^2 - y^2 = a^2$ cut off by a chord KL, parallel to OY at a distance $2a$ from OY. Show that the volume of the solid cap formed by the revolution of KAL about OX is equal to the volume of a sphere of radius a.

Show also that if the figure be rotated about the axis OY, the solid generated by the revolution of the figure MKALN has half the volume of the cylinder generated by the revolution of the rectangle MKLN. (Q.P.M.S.T.)

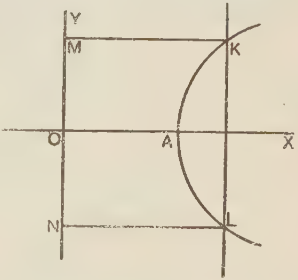


FIG. 112.

101. The curved surface of a symmetrical cask with flat ends is formed by the rotation of an arc of a parabola about a line perpendicular to its axis. Show that, if a is the radius of each end, b the radius of the maximum section, and c the height, the volume is

$$\tfrac{1}{15}\pi c(3a^2 + 4ab + 8b^2). \qquad \text{(Trinity.)}$$

102. A cube with an edge of 2 feet is held under water with one edge on the surface of the water and a face through that edge making an angle of 30° with the water surface. Find the total pressure on each face of the cube. (Q.P.M.S.T.)

103. A heavy uniform circular disc, of diameter 9 inches, is pivoted on a horizontal axis, perpendicular to the disc, through a point on the circumference. It is displaced through 135° from the position of equilibrium and released ; find the angular velocity and the kinetic energy (in ft.-lb. units) when the centre is at the lowest point of its path. (Q.P.M.S.T.)

104. If the velocity v of a moving point is expressed in terms of the time t, how would you apply the calculus to find the space described s, and the acceleration ?

Having given the relation $v = at^2 + 2bt + c$, find expressions for the space described and the acceleration at any time t.

The units being the second and the centimetre, a being equal to -1, b to 16, c to 8, and the value of s when t is zero being 22, find the space described, *i.e.* distance from the starting point and the acceleration at time 30 seconds. (Army.)

105. One cubic foot of air at a temperature of 500° C. absolute is expanded isothermally from a pressure of 120 lb. per sq. in. to twice the initial volume. It is then expanded adiabatically to three times the initial volume. Find the pressure at the end of each stage, and calculate graphically or otherwise the work done and the heat units taken in during each stage, taking the mechanical equivalent of one thermal unit as 1400 ft.-lbs. and the equation for adiabatic expansion of air as $p \times v^{1 \cdot 4} = \text{constant}$.

CHAPTER XI.

DIFFERENTIATION AND INTEGRATION OF TRIGONOMETRICAL FUNCTIONS.

1. We proceed to deal with the Differentiation and Integration of the Trigonometrical Functions.

To find the gradient at the origin of the curve $y = \sin x$ where x is measured in radians.

Let $\quad ON = \delta x.$

$\therefore \; PN = \sin \delta x.$

Hence the gradient of the line OPL is $\frac{PN}{ON}$, *i.e.* $\frac{\sin \delta x}{\delta x}$.

We have to find the limiting value of $\frac{\sin \delta x}{\delta x}$ as δx tends towards the limit zero by repeated bisection of ON or otherwise.

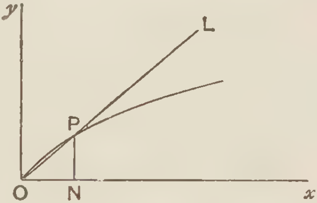


FIG. 113.

Take a circle of radius a, so that the two radii OR and OS are inclined to one another at an angle of δx radians. Plainly the ratio of the area of the triangle ORS to the area of the sector ROSU tends to become more and more equal to unity as OS tends to move closer and closer to OR by repeated bisection of the angle SOR or otherwise ....(2)

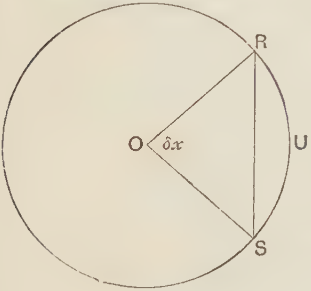


FIG. 114.

Now, the area of the triangle

$$ROS = \tfrac{1}{2}a^2 \sin \delta x$$

and the area of the sector ROSU $=\frac{1}{2}a^2\delta x$.

$$\therefore \frac{\text{area of the triangle ROS}}{\text{area of the sector ROSU}}=\frac{\sin\delta x}{\delta x}. \quad\ldots\ldots\ldots\ldots\ldots(3)$$

Hence $\dfrac{\sin\delta x}{\delta x}$ tends towards the limit unity as δx tends towards zero, since the left-hand side of (3) has been shown above to tend towards the limit unity by (2). ....(4)

Hence by (1) and (4) we see that *the gradient of the tangent at the origin to the curve $y=\sin x$ is unity, x being measured in radians.*

Corollary. *Plainly* $\dfrac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}$ *tends towards the limit unity as δx tends towards zero.*

This is plain from the conception of δx tending towards the limit zero by the method of "repeated bisection."

2. Notation.

We write the fact that $\dfrac{\sin\delta x}{\delta x}$ tends towards the limit unity as δx tends towards zero in the form

$$\text{Limit}_{\delta x\to 0}\frac{\sin\delta x}{\delta x}=1 \quad\text{or}\quad \text{Lt}_{\delta x\to 0}\frac{\sin\delta x}{\delta x}=1.$$

3. To find the differential coefficient of sin x where x is measured in radians.

As usual, let

$$y=\sin x.$$

$$y+\delta y=\sin(x+\delta x).$$

$$\therefore\ \delta y=\sin(x+\delta x)-\sin x.$$

$$\therefore\ \delta y=2\cos\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}.$$

$$\therefore\ \frac{\delta y}{\delta x}=2\cos\left(x+\frac{\delta x}{2}\right)\frac{\sin\frac{\delta x}{2}}{\delta x}.$$

$$\therefore \frac{\delta y}{\delta x} = \cos\left(x + \frac{\delta x}{2}\right) \times \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now $$\text{Lt}_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) = \cos x$$

and $$\text{Lt}_{\delta x \to 0} \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} = 1.$$

Hence, proceeding to the limit in the case of (1) when δx becomes evanescent by means of "repeated bisection" or otherwise, we see that

$$\frac{dy}{dx} = \cos x \times 1,$$

$$\textit{i.e.} \quad \frac{dy}{dx} = \cos x.$$

Corollary I. $\int \cos x \, dx = \sin x$ (x in radians).

Corollary II. *To find the differential coefficient of* $\sin x$ *where* x *is measured in degrees.*

Let x degrees $= z$ radians.

$$\therefore \frac{x}{180} = \frac{z}{\pi}.$$

$$\therefore \frac{d \sin x}{dx} = \frac{d}{dx} \sin z \quad (z \text{ in radians})$$

$$= \frac{dz}{dx} \cdot \frac{d \sin z}{dz}$$

$$= \frac{\pi}{180} \cos z$$

$$\therefore \frac{d \sin x}{dx} = \frac{\pi}{180} \cos x \quad (x \text{ in degrees}).$$

Corollary III. $\int \cos x\,dx = \frac{180}{\pi}\sin x$ (x in degrees).

N.B.—$\frac{\pi}{180} = 0{\cdot}01745$. (Mnemonic: 1745 is date of Second Jacobite Rebellion.)

4. Aliter.

The following method of looking at the problem of differentiating $\sin x$ according to the units used may help to shed further light on the question.

Let P_1, P_2, P_3, etc., be points on the curve whose abscissae are x_1, x_2, x_3, etc. Let Q_1, Q_2, Q_3, etc., be points on the curve near to P_1, P_2, P_3, etc., whose abscissae are x_1+h, x_2+h, x_3+h, etc., respectively. Then the gradients of the chords P_1Q_1, P_2Q_2, P_3Q_3, etc., are respectively,

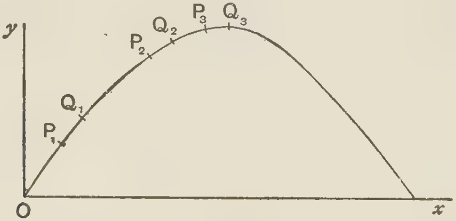


FIG. 115.

$$\frac{\sin(x_1+h)-\sin x}{h},\quad \frac{\sin(x_2+h)-\sin x_2}{h},\quad \frac{\sin(x_3+h)-\sin x_3}{h},\ \text{etc.}$$

i.e.

$$\frac{2\cos\left(x_1+\frac{h}{2}\right)\sin\frac{h}{2}}{h},\quad \frac{2\cos\left(x_2+\frac{h}{2}\right)\sin\frac{h}{2}}{h},\quad \frac{\cos\left(x_3+\frac{h}{2}\right)\sin\frac{h}{2}}{h},\ \text{etc.}$$

Hence

$$\frac{\text{radient of chord } P_1Q_1}{\cos\left(x_1+\frac{h}{2}\right)} = \frac{\text{gradient of chord } P_2Q_2}{\cos\left(x_2+\frac{h}{2}\right)} = \frac{\text{gradient of chord } P_3Q_3}{\cos\left(x_3+\frac{h}{2}\right)},\ \text{etc.,}$$

since each $= \frac{2\sin\frac{h}{2}}{h}$.

Hence, proceeding to the limit when Q_1, Q_2, Q_3, etc., move up to coincidence with P_1, P_2, P_3, etc., respectively, we see that

$$\frac{\text{gradient of tangent at } x_1}{\cos x_1} = \frac{\text{gradient of tangent at } x_2}{\cos x_2} = \frac{\text{gradient of tangent at } x_3}{\cos x_3}, \text{ etc.,}$$

whatever units for x are used, wherewith to trace the graph of $y = \sin x$.

To find this constant value for $\dfrac{\text{gradient of tangent at } x}{\cos x}$, we find it for some particular point on the curve where it is easy to calculate, for example the origin. When x is measured in radians, this constant value is $+1$, as shown above in § 1, and when x is measured in degrees it is $\dfrac{\pi}{180}$.

Thus:

$$\frac{\text{gradient of } \sin x \text{ at } x}{\cos x} \equiv \frac{d \sin x}{dx} = +1 \quad (x \text{ measured in radians})$$
$$= \frac{\pi}{180} \quad (x \text{ measured in degrees}).$$

Example 1. *Calculate* $\dfrac{d \sin x}{dx}$ *when* $x = \dfrac{\pi}{3}$.

We have $$\frac{d \sin x}{dx} = \cos x.$$

$$\therefore \frac{d \sin x}{dx} = \tfrac{1}{2} \quad \left(\text{when } x = \frac{\pi}{3}\right).$$

Example 2. *Differentiate* $y = \sin (5x + 7)^\circ$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d \sin (5x+7)^\circ}{dx} \\ &= \frac{d \sin (5x+7)^\circ}{d(5x+7)} \times \frac{d(5x+7)}{dx} \\ &= 0{\cdot}01745 \times \cos (5x+7)^\circ \times 5 \\ &\equiv 0{\cdot}0873 \cos (5x+7)^\circ. \end{aligned}$$

Example 3. *The range* R *ft. of a gun projecting a bullet with velocity*

V f.s. and elevated at an angle θ° from the horizontal (air resistance being neglected) is given approximately by the formula

$$R=\frac{V^2\sin 2\theta^\circ}{32}.$$

The gun fires with a muzzle velocity of 1000 *f.s., and is set at an elevation of* 30°. *Calculate the rate at which the range is increasing* (1) *per extra "ft. per sec. of muzzle velocity,"* (2) *per extra degree of elevation.*

In the first case we want $\frac{dR}{dV}$ ft. (per ft. per sec. of increase of muzzle velocity).

In the second case we want $\frac{dR}{d\theta}$ ft. (per degree increase in the elevation of the gun).

$$\text{Now}\qquad \frac{dR}{dV}=\frac{V\sin 2\theta^\circ}{16}=\frac{1000\sin 60^\circ}{16}=\frac{1000\times 0{\cdot}8660}{16}=\frac{433}{8}=54.$$

∴ the range increases 54 feet for every "ft. per sec." increase of the muzzle velocity.

$$\text{Again,}\qquad \frac{dR}{d\theta}=\frac{V^2\cos 2\theta}{16}\times\frac{\pi}{180}=\frac{1000^2\times\frac{1}{2}}{16}\times 0{\cdot}01745$$

$$=\frac{1000\times 17{\cdot}45}{32}=\frac{17450}{32}=545.$$

∴ the range is increasing at the rate of 5·45 feet for every $\frac{1}{100}$th degree increase in the elevation, when the gun is set at 30°.

Example 4. *Differentiate* $y=\sin^3 x$ (*x in radians*).

$$\frac{dy}{dx}=\frac{d\sin^3 x}{dx}$$

$$=\frac{d\sin^3 x}{d\sin x}\times\frac{d\sin x}{dx}$$

$$=3\sin^2 x\times\cos x$$

$$=3\sin^2 x\cos x.$$

Example 5. *The arms of a "Flip-Flap" are* 30 *yards long. They ascend with a uniform angular velocity from the horizontal to the vertical in* 10 *minutes. Find the rate at which the people in the cage are ascending in a vertical direction when the arm is half way up.*

[*N.B.*—If y yards be the height of a body above the ground t minutes after the start of the motion, its vertical velocity is $\frac{dy}{dt}$ yards per minute, even though the body be not going directly upwards.]

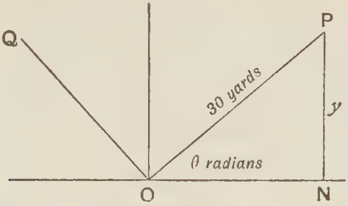


FIG. 116.

If y yards be the height of P (*i.e.* the cage) above the ground in t minutes, we require to find $\frac{dy}{dt}$ yards per minute. From the figure

$$y = 30 \sin \theta.$$

Also

$$\frac{d\theta}{dt} = \frac{\pi}{20} \text{ radians per min.}$$

$$\therefore \frac{dy}{dt} = \frac{dy}{d\theta} \times \frac{d\theta}{dt}$$

$$= 30 \cos \theta \times \frac{\pi}{20} \text{ yards per min.}$$

$$= 3 \times \frac{1}{\sqrt{2}} \times \frac{\pi}{2} \text{ yards per min. (when half way up)}$$

$$= \frac{3\sqrt{2} \times \pi}{4} \qquad \text{"} \qquad \text{"} \qquad \text{"}$$

$$= 3{\cdot}33 \text{ yards per min. (when half way up.)}$$

5. To differentiate cos x where x is measured in radians.

We have

$$y = \cos x.$$

$$\therefore y + \delta y = \cos (x + \delta x).$$

$$\therefore \delta y = \cos (x + \delta x) - \cos x.$$

$$\therefore \delta y = -2 \sin \left(x + \frac{\delta x}{2}\right) \sin \frac{\delta x}{2} \cdot$$

$$\therefore \frac{\delta y}{\delta x} = -\sin\left(x+\frac{\delta x}{2}\right)\times\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}.$$

Now $$\text{Lt}_{\delta x\to 0}\sin\left(x+\frac{\delta x}{2}\right)=\sin x$$

and $$\text{Lt}_{\delta x\to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}=1.$$

Hence, proceeding to the limit, as δx becomes indefinitely small we see that

$$\frac{dy}{dx}=-\sin x\times 1.$$

$$\therefore \frac{dy}{dx}=-\sin x.$$

Corollary I. If x be measured in degrees,

$$\frac{d\cos x}{dx}=-\frac{\pi}{180}\sin x=-0{\cdot}01745\sin x.$$

Corollary II. $\int\sin x\,dx=-\cos x$ (x in radians)

$$=-\frac{180}{\pi}\cos x \quad (x \text{ in degrees}).$$

6. To differentiate tan x where x is measured in radians.

We have $y=\tan x.$

$$\therefore y+\delta y=\tan(x+\delta x).$$

$$\therefore \delta y=\tan(x+\delta x)-\tan x$$

$$=\frac{\sin(x+\delta x)}{\cos(x+\delta x)}-\frac{\sin x}{\cos x}$$

$$=\frac{\sin(x+\delta x)\cos x-\cos(x+\delta x)\sin x}{\cos x\cos(x+\delta x)}$$

$$= \frac{\sin\left(\overline{x+\delta x}-x\right)}{\cos x \cos (x+\delta x)}$$

$$= \frac{\sin \delta x}{\cos x \cos (x+\delta x)}.$$

$$\therefore \frac{\delta y}{\delta x} = \frac{1}{\cos x \cos (x+\delta x)} \times \frac{\sin \delta x}{\delta x}.$$

Hence, proceeding to the limit, as δx becomes indefinitely small we see that

$$\frac{dy}{dx} = \frac{1}{\cos^2 x} \times 1,$$

$$i.e. \quad \frac{dy}{dx} = \sec^2 x.$$

Corollary I. If x be measured in degrees,

$$\frac{d \tan x}{dx} = \frac{\pi}{180} \sec^2 x = 0{\cdot}01745 \sec^2 x.$$

Corollary II. $\int \sec^2 x\, dx = \tan x \quad (x \text{ in radians})$

$$= \frac{180}{\pi} \tan x \quad (x \text{ in degrees}).$$

7. To differentiate cosec x where x is measured in radians.

We have $y = \operatorname{cosec} x.$

$$\therefore y + \delta y = \operatorname{cosec} (x+\delta x).$$

$$\therefore \delta y = \operatorname{cosec} (x+\delta x) - \operatorname{cosec} x$$

$$= \frac{1}{\sin (x+\delta x)} - \frac{1}{\sin x}$$

$$= -\frac{\sin (x+\delta x) - \sin x}{\sin x \sin (x+\delta x)}$$

$$= -2\frac{\cos\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\sin x \sin (x+\delta x)}.$$

$$\therefore \frac{\delta y}{\delta x} = -\frac{\cos\left(x+\frac{\delta x}{2}\right)}{\sin x \sin (x+\delta x)} \times \frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}.$$

Hence, proceeding to the limit, as δx becomes indefinitely small we see that

$$\therefore \frac{dy}{dx} = -\frac{\cos x}{\sin^2 x} \times 1,$$

$$i.e. \frac{dy}{dx} = -\operatorname{cosec} x \cot x \text{ (in another form).}$$

Corollary. If x be measured in degrees,

$$\frac{d \operatorname{cosec} x}{dx} = -\frac{\pi}{180} \operatorname{cosec} x \cot x = -0{\cdot}01745 \operatorname{cosec} x \cot x.$$

8. To differentiate sec x, where x is measured in radians.

We have $y = \sec x.$

$$\therefore y + \delta y = \sec (x + \delta x).$$

$$\begin{aligned} \therefore \delta y &= \sec (x + \delta x) - \sec x \\ &= \frac{1}{\cos (x + \delta x)} - \frac{1}{\cos x} \\ &= \frac{\cos x - \cos (x + \delta x)}{\cos x \cos (x + \delta x)} \\ &= 2\frac{\sin \left(x + \frac{\delta x}{2}\right) \sin \frac{\delta x}{2}}{\cos x \cos (x + \delta x)}. \end{aligned}$$

$$\therefore \frac{\delta y}{\delta x} = \frac{\sin \left(x + \frac{\delta x}{2}\right)}{\cos x \cos (x + \delta x)} \times \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}.$$

Hence, proceeding to the limit, as δx becomes indefinitely small we see that

$$\therefore \frac{dy}{dx} = \frac{\sin x}{\cos^2 x} \times 1,$$

$$i.e. \frac{dy}{dx} = \sec x \tan x \text{ (in another form).}$$

Corollary. If x be measured in degrees,

$$\frac{d \sec x}{dx} = \frac{\pi}{180} \sec x \tan x = 0{\cdot}01745 \sec x \tan x.$$

9. To differentiate cot x, where x is measured in radians.

We have $\qquad y = \cot x.$

$$\therefore\ y + \delta y = \cot (x + \delta x).$$

$$\therefore\ \delta y = \cot (x + \delta x) - \cot x$$

$$= \frac{\cos (x + \delta x)}{\sin (x + \delta x)} - \frac{\cos x}{\sin x}$$

$$= \frac{\sin x \cos (x + \delta x) - \cos x \sin (x + \delta x)}{\sin x \sin (x + \delta x)}$$

$$= -\frac{\sin \delta x}{\sin x \sin (x + \delta x)}$$

$$\therefore\ \frac{\delta y}{\delta x} = -\frac{1}{\sin x \sin (x + \delta x)} \times \frac{\sin \delta x}{\delta x}.$$

Hence, proceeding to the limit, as δx becomes indefinitely small we see that

$$\therefore\ \frac{dy}{dx} = -\frac{1}{\sin^2 x} \times 1.$$

$$i.e.\ \frac{dy}{dx} = -\operatorname{cosec}^2 x.$$

Corollary I. If x be measured in degrees,

$$\frac{d \cot x}{dx} = -\frac{\pi}{180} \operatorname{cosec}^2 x = -0{\cdot}01745 \operatorname{cosec}^2 x.$$

Corollary II. $\int \operatorname{cosec}^2 x\, dx = -\cot x \quad (x \text{ in radians})$

$$= -\frac{180}{\pi} \cot x \quad (x \text{ in degrees}).$$

Example 1. *Trace the graph of* $y = 3 \tan x^\circ + 4 \cot x^\circ$ *from* 0° *to* 360°, *and find its turning values.*

We have $\qquad \frac{dy}{dx} = \frac{\pi}{180}(3 \sec^2 x^\circ - 4 \operatorname{cosec}^2 x^\circ).$

Hence $\frac{dy}{dx}$ vanishes when

$$3 \sec^2 x = 4 \operatorname{cosec}^2 x,$$

$$i.e.\ \tan x = \pm\sqrt{\frac{4}{3}} = \pm\sqrt{1{\cdot}333}$$

$$= \pm 1{\cdot}154.$$

$$\therefore\ x = 49°\ 6',\ \ 130°\ 54',\ \ 229°\ 6',\ \ 310°\ 54'.$$

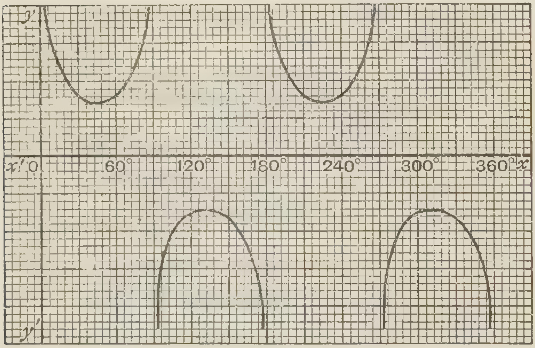


FIG. 117.

The graph shows that the values of x correspond to a minimum, maximum, minimum, maximum respectively.

Example 2. XY *is a slider moving freely on the rod* AB. PQ *is another slider such that the fixed pin* E *moves in the slot of* PQ. *If* XY *has got a velocity of* 40 *ft. per sec. towards* H *when the inclination of* YP *to* YB *is* 30°, *calculate the angular velocity of* YP, *the distance* EH *being* 2 *ft.*

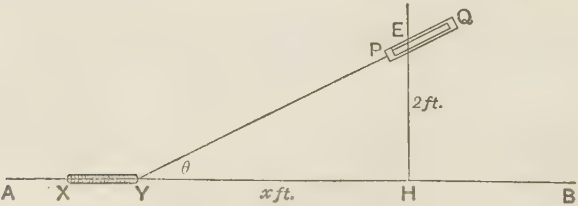


FIG. 118.

Let $\quad YH = x$ ft.

$$\therefore\ YH = HE \cot\theta.$$

$$\therefore\ x = 2\cot\theta.$$

$$\therefore\ \frac{dx}{dt} = 2 \times \frac{\pi}{180} \times (-\operatorname{cosec}^2\theta)\frac{d\theta}{dt}.$$

$$\therefore\ \frac{dx}{dt} = -\frac{\pi}{90}\operatorname{cosec}^2\theta\frac{d\theta}{dt}.$$

$$\therefore\ \frac{d\theta}{dt} = -\frac{90\sin^2\theta}{\pi}\frac{dx}{dt}.$$

$$\therefore\ \frac{d\theta}{dt} = -\frac{90\sin^2 30^\circ}{\pi} \times (-40)$$

$$= +\frac{900}{3{\cdot}1416} \text{ degrees per second}$$

$$= 286{\cdot}5 \text{ degrees per second.}$$

Example 3. *Find the rhombus of least perimeter that can be circumscribed to a circle.*

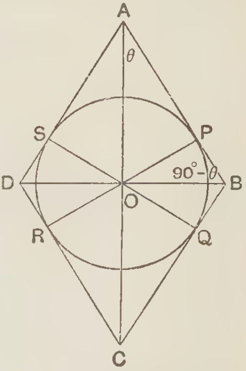


FIG. 119.

The figure explains itself.

Plainly $AP = OP\cot\theta$ (θ in radians)

and $PB = OP\tan\theta$.

$$\therefore\ AB = OP(\cot\theta + \tan\theta).$$

Hence, if p denote the perimeter and a denote the radius,

$$\therefore\ p = 4a(\cot\theta + \tan\theta).$$

If we graph p against θ we find that the graph possesses a turning-point as θ varies from 0 to $\frac{\pi}{2}$, which are the lower and upper limits to the possible variations of θ as defined by the above figure. Also the graph shows that this turning-point is a minimum.

Now, $$\frac{dp}{d\theta} = 4a(-\operatorname{cosec}^2\theta + \sec^2\theta).$$

Hence p acquires its minimum value when $-\operatorname{cosec}^2\theta + \sec^2\theta = 0$,

i.e. when $\tan^2\theta = 1$,

i.e. when $\theta=\frac{\pi}{4}$,

i.e. when the rhombus is a square.

Example 4. *Ox and Oy are two smooth rods fixed at right angles to one another.* PQ *is a rod* 10 *cm. long which moves so that* P *is on* O*x and* Q *is on* O*y. Find the ratio of the velocity of* P *to the velocity of* Q.

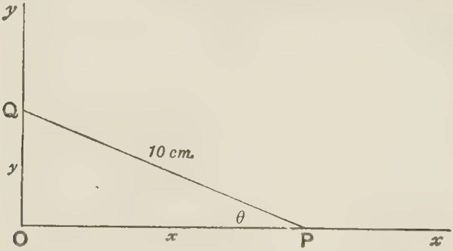


FIG. 120.

We have $x=10\cos\theta$ (θ in radians)

and $y=10\sin\theta$.

$$\therefore \frac{dx}{dt}=-10\sin\theta\frac{d\theta}{dt}$$

and
$$\frac{dy}{dt}=10\cos\theta\frac{d\theta}{dt}.$$

$$\therefore \frac{\frac{dx}{dt}}{\frac{dy}{dt}}=-\tan\theta.$$

Hence the ratio of their velocities in the position defined by θ is $-\tan\theta$.

Example 5. *If* $x=a\sec\phi$ *and* $y=b\operatorname{cosec}\phi$, *find* $\frac{dy}{dx}$ *in terms of* ϕ *(*ϕ *being in degrees).*

We have

$$\frac{dy}{dx}=\frac{\frac{dy}{d\phi}}{\frac{dx}{d\phi}}=\frac{0\cdot01745\times(-b\operatorname{cosec}\phi\cot\phi)}{0\cdot01745\times(a\sec\phi\tan\phi)}=-\frac{b}{a}\cot^3\phi.$$

Example 6. *A man stands two miles from a railway and watches with a telescope a train going at 60 miles an hour. How fast is his telescope moving angularly?*

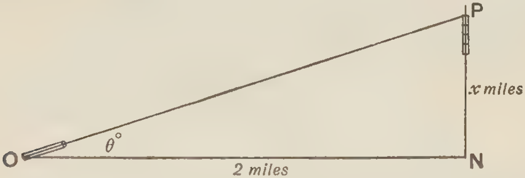


FIG. 121.

Let O be the position of the man, and ON perpendicular to the railway. Let P be the train.

Then $$x = 2\tan\theta.$$

$$\therefore\ \frac{dx}{dt} = 2 \times \frac{\pi}{180}\sec^2\theta\,\frac{d\theta}{dt}.$$

$$\therefore\ \frac{d\theta}{dt} = \frac{90}{\pi}\cos^2\theta\,.\,\frac{dx}{dt}$$

$$= \frac{90}{\pi}\cos^2\theta \times 60 \text{ degrees per hour}$$

$$= \frac{90}{\pi}\cos^2\theta \text{ degrees per minute.}$$

Hence, when the train is at N, $\cos^2\theta = 1$ and $\frac{d\theta}{dt} = 28{\cdot}7$ degrees per minute (approx.).

As the train proceeds, he has to turn his telescope more and more slowly. Thus, when $\theta = 45°$, his rate of turning is 14·3 degrees per minute.

10. We shall now give some examples of integration in the case of the trigonometric functions. We remember that

$$\int \sin x\,dx = -\cos x \quad (x \text{ in radians})$$

$$= -\frac{180}{\pi}\cos x \quad (x \text{ in degrees})$$

and

$$\int \cos x\,dx = \sin x \quad (x \text{ in radians})$$

$$= \frac{180}{\pi}\sin x \quad (x \text{ in degrees}).$$

$$\int \sec^2 x\,dx = \tan x \quad (x \text{ in radians})$$

$$= \frac{180}{\pi}\tan x \quad (x \text{ in degrees}).$$

Example 1. *Evaluate* $\int_0^{\frac{\pi}{6}}(3\sin x+4\cos x)dx$.

$$\begin{aligned}\int_0^{\frac{\pi}{6}}(3\sin x+4\cos x)dx&=\Big[-3\cos x+4\sin x\Big]_0^{\frac{\pi}{6}}\\&=[-3(0{\cdot}8660-1)+4(0{\cdot}5000-0)]\\&=[3(0{\cdot}1340)+2{\cdot}0000]\\&=2{\cdot}4020.\end{aligned}$$

Example 2. *Evaluate* $\int_0^{36^\circ}(5\cos x-7\sin x)dx$.

$$\begin{aligned}\int^{36^\circ}(5\cos x-7\sin x)dx&=\frac{180}{\pi}\Big[5\sin x+7\cos x\Big]_{0^\circ}^{36^\circ}\\&=\frac{180}{\pi}[5(0{\cdot}5878-0)+7(0{\cdot}8090-1)]\\&=\frac{180}{\pi}[2{\cdot}9390-1{\cdot}3370]\\&=\frac{180}{\pi}\times 1{\cdot}6020\\&=91{\cdot}77\end{aligned}$$

EXAMPLES XI.

1. Differentiate $2\sin x$, $4\sin x+3$, $\sin(x+2)$, $7+\sin(x+5)$, $\sin 5x$, $\sin(6x+5)$, $4+\sin(7x+9)$, x being measured in radians throughout.

2 If $y=\sin x$, find $\frac{dy}{dx}$ when $x=3$ radians.

3. If $y=7+\sin(3x-1)$, find $\frac{dy}{dx}$ when $x=0{\cdot}5$ radians.

4. If $y=4\sin(4-3x)$, find $\frac{dy}{dx}$ when $x=2{\cdot}5$ radians.

5. Differentiate $\sin 3x$, $4\sin x+7$, $8-5\sin x$, $4\sin 3x$, $9-4\sin 5x$, $3\sin(2x+1)$, $7-6\sin(4x-5)$, $9-7\sin(5-3x)$, x being measured in degrees throughout.

6. Differentiate

$3\cos x$, $4\cos(3x-4)$, $7-5\cos(7x-2)$, $2+3\cos(2x+1)+4\sin(2x-1)$, x being measured in radians throughout.

7. If $y=\sin x+\cos x$, find $\frac{dy}{dx}$ when $x=37°$.

8. If $y = \sin 2x + \cos 2x$, prove that $\frac{dy}{dx} + 2y = 4\cos 2x$.

9. Draw the graph of $y = 2\sin x - 3\cos x$ between $x = 0$ and $x = 360°$, and from your graph find a *rough check* to the positions and nature of the turning values. Thereafter find $\frac{dy}{dx}$, and hence find accurately the positions of the turning values.

10. If $y = \sin px$, prove that $\frac{d^2y}{dx^2} + p^2y = 0$.

11. If $y = \mathrm{A}\sin qx + \mathrm{B}\cos qx$, prove that $\frac{d^2y}{dx^2} + q^2y = 0$.

12. If $y = a\sin(nx + \alpha)$, prove that $\frac{d^2y}{dx^2} + n^2y = 0$.

13. A particle moves in a straight line so that its distance x cm. from a fixed point after t seconds is given by $x = a\cos(\omega t + \epsilon)$. Find its velocity and acceleration after t seconds. When is its velocity a maximum, and when a minimum?

14. Ox and Oy are two fixed lines at right angles to one another. OP is a variable line of length 10 cm., lying between Ox and Oy and in the same plane with them. PM is perpendicular to Ox, and PN is perpendicular to Oy.

If the angle PO$x = \theta$ radians, express PM + PN in terms of θ. Plot the graph of PM + PN against θ, and find the position for which PM + PN is greatest. Confirm by differentiation.

15. In question 14, express the area of the rectangle OMPN in terms of θ, and hence find when the area of this rectangle is greatest.

16. A door APQB 4 ft. wide is kept closed by a string 16 ft. long fastened to the door at P, passing over a pulley at C, and supporting the weight W. In the figure the door is represented as standing ajar, having been turned through an angle of 2θ degrees. If x ft. be the length of CW, find the length of CP in terms of θ.

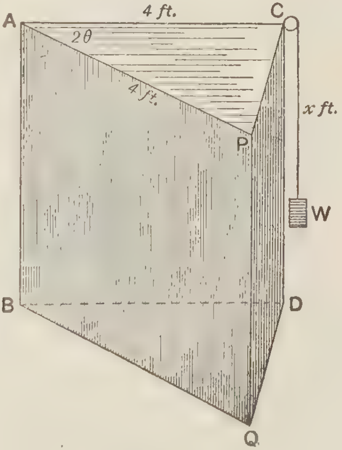


FIG. 122.

Hence find the relation between $\frac{dx}{dt}$ and $\frac{d\theta}{dt}$.

If the door is being opened at the uniform rate of 45° per second, find the rate at which the weight W is ascending, (first) when the door is ajar 45°, and (second) when the door is ajar 90°.

17. The jib of a crane is being elevated at the rate of 1° per second. Find the rate at which the load is being raised in terms of θ, where θ degrees is the inclination of the jib to the horizontal, the length of the jib being 20 ft.

What is the inclination of the jib when the weight is being raised at the rate of 0·2 ft. per sec.?

18. In question 17, if the jib is being raised with a uniform angular velocity of ω degrees per second, and if v be the vertical velocity with which the weight is rising corresponding to the angle θ, calculate v in terms of ω and θ.

19. In the figure, OP is the jib of a crane 15 ft. long. The sun's elevation is 45°, and OQ is the shadow cast by OP. As the jib OP is raised at the rate of 2° per second, find the rate at which the end Q of the shadow is receding from O, when the inclination of the jib to the horizontal is 30°.

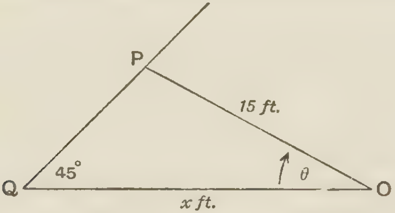


FIG. 123.

Express the velocity v of Q in terms of θ.

If the jib be rising with a uniform angular velocity of ω degrees per second, express v in terms of ω and θ.

20. Prove that $y=\tan x$ (x in radians) is a solution of the differential equation

$$\frac{dy}{dx}-y^2-1=0.$$

21. Prove that $y=\cot x$ (x in radians) is a solution of the differential equation

$$\frac{dy}{dx}+y^2+1=0.$$

22. If $y=\tan x$, find $\frac{d^2y}{dx^2}$ in terms of x (x in radians).

23. If $y=\cot x$, find $\frac{d^2y}{dx^2}$ in terms of x (x in degrees).

24. Prove that $y=\sec x$ is a solution of the differential equation

$$\frac{d^2y}{dx^2}=\tan x\frac{dy}{dx}+y^3.$$

25. Differentiate $3\tan x$, $\tan 3x$, $4-5\tan 2x$, $4\cot x$, $\cot 7x$, $3-2\tan x+7\cot x$, $3\tan 2x-2\cot 3x$, x being measured in radians throughout.

26. If $y=\tan x+\cot x$, find $\frac{dy}{dx}$ when $x=1{\cdot}5$ radians.

27. If $y=10\tan\theta$, find θ given that $\frac{dy}{dt}=7$ and $\frac{d\theta}{dt}=0{\cdot}2$, θ being measured in radians.

28. If $y=\tan x+\cot x$, find $\frac{dy}{dx}$ when $x=35°$.

29. If $y=\tan 2x-\cot 2x$, find $\frac{dy}{dx}$ when $x=20°$.

30. Ox and Oy are two lines at right angles to one another. A circle of radius a cm. is described having O as centre. The tangent at the point P on the circle meets Ox in Q and Oy in R. If $POx=\theta$ radians, express OQ + OR in terms of θ.

Hence find when OQ + OR is a minimum.

31. If $x=\tan\theta$ and $y=\cot\theta$, find $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$, and hence find $\frac{dy}{dx}$ in terms of θ.

32. Differentiate $\sec(x+7)$, $2\sec(3x+4)$, $3-4\operatorname{cosec}(4x+5)$, $5+4\sec x-3\operatorname{cosec} x$, x being measured in radians throughout.

33. If $y=\sec\phi$ and $x=\operatorname{cosec}\phi$, find $\frac{dy}{dx}$ in terms of ϕ.

34. A rhombus is made of four rods each of 10 cm. length and freely jointed at their extremities. Find the greatest area that the four rods can be made to enclose.

35. What is the maximum area of a parallelogram whose sides are 11 cm. and 9 cm.?

36. Find the area of the rhombus of minimum area which can be circumscribed about a circle of radius 4 inches.

37. A balloon rises vertically at the uniform rate of 6 miles per hour from a spot 1 mile distant from an observer with a telescope. Find the rate at which he must rotate the telescope to keep the balloon in the field of view when the balloon is 3 miles high.

38. In question 37, at what rate is the distance between the balloon and the observer increasing when the balloon is 3 miles high?

39. If a balloon descends vertically from a height of 2 miles so that the line joining it to an observer distant 2 miles from its landing-point rotates at the uniform rate of 3° per min., what is the speed of the balloon when $\frac{1}{2}$ mile high?

40. An aeroplane is sighted flying horizontally at a height of $\frac{1}{2}$ mile and at the rate of 80 miles per hour. If the telescope at that moment makes an angle of 70° with the horizon, at what rate must the telescope be rotated to keep the aeroplane in the field of view? The observer and the line of flight are in the same vertical plane.

41. An avenue consists of two rows of trees, the road between them being 40 ft. wide. Find the rate at which the angle increases which an opposite pair subtend at the eye of a cyclist riding at 10 miles per hour down the avenue when he is 100 yards away.

42. Two trains, one running at 40 miles per hour, the other at 25 miles per hour, in opposite directions, have their engines level as they pass an observer $\frac{1}{4}$ mile away from the tracks. Find the rate at which the angle increases which the engines subtend at the observer's eye 1 minute after they have passed him.

43. A pair of compasses very stiff in the joint are laid across a wooden cylinder (*e.g.* a rolling-pin) of diameter 2 inches, and pressed slowly down so that the joint moves vertically at the rate of 1 inch per minute. Find the rate at which the angle θ between the legs of the compasses is increasing when $\theta = 10°, 20°, 30°$.

44. A man is standing on a platform 10 ft. from a lamp which is carried by a non-stop express running at 60 miles per hour. What is the rate at which his shadow on the platform rotates when the lamp is exactly opposite to him, and also 1 sec. later?

45. If $V = RC + L\frac{dC}{dt}$, and if $C = 100 \sin 600t$, R being 2 and L being 0·0005, find V. What is the lag of C in degrees behind V.

(Bd. of Educ.)

46. The driving-wheel of a locomotive is 6 ft. in diameter, and the connecting-rod, which is 3 ft. long, is connected to a crank 1 ft. in length. The engine is running at 30 miles per hour. Find the velocity of the piston when the crank is perpendicular to the line of travel of the piston.

47. Draw the graphs of $y=\sin x$ and $y=\cos x$ between 0 and $\frac{\pi}{2}$, and let a line parallel to the y axis cut them in P and Q respectively. Prove that the subnormals of P and Q are of equal lengths.

48. A circular disc, area 10 sq. in., is distant $2\frac{1}{2}$ ft. from a wall and parallel to the wall. A light is moved in a straight line passing through the centre of the disc and perpendicular to it at the rate of 5 ft. per sec. Find the rate of growth of the area of the shadow of the disc on the wall when the light is distant 20 ft. from the wall.

49. The arm of a signal is 15 ft. from the ground when "at danger," and the length of the arm is 3 ft. On a certain occasion the sun is in the same vertical plane as that swept out by the arm, and its altitude is 35°. If the arm revolve with uniform angular velocity ω radians per sec., what is the rate at which the end of the shadow cast by the arm moves on the ground when the arm makes an angle of 30° with the horizon.

50. Sketch the graph of $y=\sin x$ (x in radians) from $x=0$ to $x=\pi$, and find the area intercepted between the curve and the axis of x.

51. Sketch the graph of $y=\cos x$ (x in degrees) from $x=0°$ to $x=90°$, and find the area intercepted between the curve and the axes of x and y.

52. Find the value of $\int_0^{\frac{\pi}{2}}(7\sin x-15\cos x)dx$ (x in radians).

53. Find the value of $\int_{0°}^{32°}(3\cos x+11\sin x)dx$ (x in degrees).

54. Find the value of $\int_0^{\frac{\pi}{3}}4\sec^2 x\,dx$ (x in radians).

55. Find the value of $\int_{25°}^{55°}\operatorname{cosec}^2 x\,dx$ (x in degrees).

56. Draw the graph of $y=\sec x\tan x$ (x in radians) from $x=0$ to $x=\frac{\pi}{3}$, and find the area enclosed between the curve, the axis of x, and the ordinate corresponding to $x=\frac{\pi}{3}$.

57. Find the value of $\int_{10°}^{72°}\operatorname{cosec} x\cot x\,dx$ (x in degrees).

CHAPTER XII.

DIFFERENTIATION OF PRODUCTS AND QUOTIENTS; PARAMETERS.

1. Frequently we have to differentiate products and quotients such as $x \sin x$, $\dfrac{x}{\cos x}$, $(x^2+3)\tan^2 x$. We shall now investigate methods of carrying out such differentiations.

2. Differentiation of a product.

Let u and v be two functions of x of the type that present themselves in practical investigations.

Then $$y = uv \quad\text{...(1)}$$

$$\therefore\ y+\delta y = (u+\delta u)(v+\delta v)$$
$$= uv + u\,\delta v + v\,\delta u + \delta u\,\delta v \quad\text{..................(2)}$$

Hence, subtracting (1) from (2),

$$\delta y = u\,\delta v + v\,\delta u + \delta u\,\delta v.$$

$$\therefore\ \frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x}\,\delta v.$$

Hence, proceeding to the limit in the usual way, we obtain

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx},$$

since the term $\dfrac{\delta u}{\delta x}\,\delta v$ tends to vanish, because δv tends to the limit zero.

3. Example 1. *Differentiate* $x \sin x$ *(*x *measured in radians).*

We have $$y = x \sin x.$$

$$\therefore\ \frac{dy}{dx} = x\frac{d\sin x}{dx} + \sin x\frac{dx}{dx}$$
$$= x\cos x + \sin x.$$

Example 2. *Differentiate* $(x+3)\tan x$ *(x measured in radians).*

We have
$$y=(x+3)\tan x.$$
$$\therefore \frac{dy}{dx}=(x+3)\frac{d\tan x}{dx}+\tan x\frac{d(x+3)}{dx}$$
$$=(x+3)\sec^2 x+\tan x.$$

Example 3. *Verify that* $\dfrac{d\sin 2\theta}{d\theta}=2\cos 2\theta$ *(θ in radians), using the product formula.*

We have
$$\sin 2\theta=2\sin\theta\cos\theta.$$
$$\therefore \frac{d\sin 2\theta}{d\theta}=2\sin\theta\frac{d\cos\theta}{d\theta}+2\cos\theta\frac{d\sin\theta}{d\theta}$$
$$=-2\sin^2\theta+2\cos^2\theta$$
$$=2(\cos^2\theta-\sin^2\theta)$$
$$=2\cos 2\theta.$$

Example 4. *Find the first positive turning value of* $x\cos x$, *x being measured in radians.*

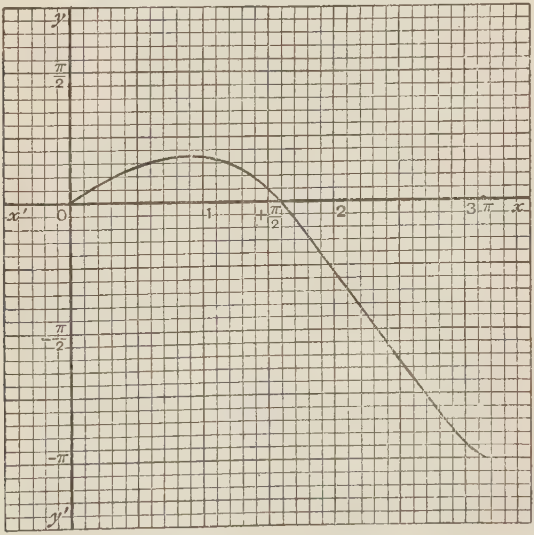


FIG. 124.

We have
$$y = x\cos x.$$
$$\therefore\ \frac{dy}{dx} = \cos x - x\sin x.$$

Hence $\frac{dy}{dx}$ vanishes when
$$\cos x = x\sin x,$$
$$i.e.\ \cot x = x,$$
which equation being solved graphically as the intersection of $y = x$ with $y = \cot x$ gives us $x = 0{\cdot}86$.

Example 5. *Find the right circular cone of greatest volume which can be inscribed in a given sphere.*

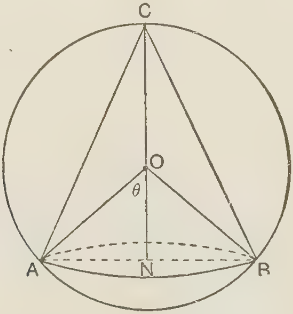


FIG. 125.

The above figure explains itself.

Let a be the radius of the sphere, and V the volume of the cone.

$$\therefore\ V = \frac{1}{3}\pi\,.\,NA^2\,.\,CN$$
$$= \frac{\pi}{3}a^3\sin^2\theta(\cos\theta + 1).$$
$$\therefore\ \frac{dV}{d\theta} = \frac{\pi a^3}{3}\left\{\sin^2\theta\frac{d}{d\theta}(\cos\theta+1) + (\cos\theta+1)\frac{d}{d\theta}\sin^2\theta\right\}$$
$$= \frac{\pi a^3}{3}\left\{-\sin^3\theta + 2\sin\theta\cos\theta(\cos\theta+1)\right\}.$$

Hence, graphing V against θ, we see that V is a maximum when $\frac{dV}{d\theta} = 0$,

i.e. when $-\sin^3\theta+2\sin\theta\cos\theta(\cos\theta+1)=0$;
i.e. when $\sin\theta(3\cos\theta-1)(\cos\theta+1)=0$ (factorising);
i.e. when $\cos\theta=\frac{1}{3}=0{\cdot}3333$ (ignoring $\cos\theta=-1$ and $\sin\theta=0$ as unsuitable solutions);
i.e. when $\theta=70°\ 30'$.

4. Differentiation of a quotient.

Let $y=\frac{u}{v}$, where u and v are functions of x of the type that present themselves in practical work. ....(1)

$$\therefore\ y+\delta y=\frac{u+\delta u}{v+\delta v}. \qquad\text{........................(2)}$$

Hence, subtracting (1) from (2),

$$\delta y=\frac{u+\delta u}{v+\delta v}-\frac{u}{v}$$
$$=\frac{v\,\delta u-u\,\delta v}{v(v+\delta v)}.$$
$$\therefore\ \frac{\delta y}{\delta x}=\frac{v\frac{\delta u}{\delta x}-u\frac{\delta v}{\delta x}}{v(v+\delta v)}.$$

Hence, proceeding to the limit in the usual way, we obtain

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}.$$

5. Example 1. *Verify by using the quotient formula that* $\frac{d}{dx}\tan x=\sec^2 x$, *$x$ being measured in radians.*

For
$$\frac{d}{dx}\tan x=\frac{d}{dx}\frac{\sin x}{\cos x}$$
$$=\frac{\cos x\times\frac{d}{dx}\sin x-\sin x\frac{d}{dx}\cos x}{\cos^2 x}$$
$$=\frac{\cos x\times\cos x-\sin x\times(-\sin x)}{\cos^2 x}$$

$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}$$

$$=\frac{1}{\cos^2 x}$$

$$=\sec^2 x.$$

Example 2. *For what values of x is $\frac{\sin x}{x}$ a maximum or minimum, x being measured in radians?*

We have $$y=\frac{\sin x}{x},$$
whose graph is of the form

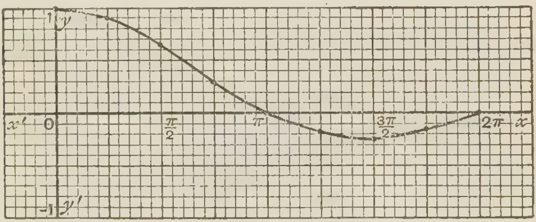


FIG. 126.

The maxima-minima are given by

$$\frac{d}{dx}\frac{\sin x}{x}=0,$$

$$i.e.\quad \frac{x\cos x-\sin x}{x^2}=0,$$

$$i.e.\text{ when }\ x\cos x-\sin x=0,$$

$$i.e.\text{ when }\ \tan x=x,$$

the solutions to which can be found graphically by reading off on squared paper the intersections of $y=\tan x$ and $y=x$.

The values of x which satisfy $x=\tan x$ are approximately 0 and 4·5, and we see from the graph of $\frac{\sin x}{x}$ that its values there are a maximum and a minimum respectively.

Example 3. *If $y=\frac{\tan x}{1+x}$, find $\frac{dy}{dx}$, where x is measured in degrees.*

We have
$$\frac{dy}{dx}=\frac{(1+x)\dfrac{d\tan x}{dx}-\tan x\dfrac{d}{dx}(1+x)}{(1+x)^2}$$
$$=\frac{(1+x)\dfrac{\pi}{180}\sec^2 x-\tan x}{(1+x)^2}$$
$$=\frac{\pi}{180}\frac{\sec^2 x}{1+x}-\frac{\tan x}{(1+x)^2}.$$

Example 4. *If* $y=\dfrac{x}{\sqrt{1+x^2}}$, *find* $\dfrac{dy}{dx}$.

$$\frac{dy}{dx}=\frac{\sqrt{1+x^2}\dfrac{dx}{dx}-x\dfrac{d}{dx}(1+x^2)^{\frac{1}{2}}}{(1+x^2)}$$
$$=\frac{\sqrt{1+x^2}-x\dfrac{d(1+x^2)^{\frac{1}{2}}}{d(1+x^2)}\times\dfrac{d(1+x^2)}{dx}}{(1+x^2)}$$
$$=\frac{\sqrt{1+x^2}-x\times\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\times 2x}{(1+x^2)}$$
$$=\frac{1}{(1+x^2)^{\frac{3}{2}}}.$$

Example 5. *A man has wire-netting a mile long, and wishes to fence in as large an area as possible in the shape of a circular enclosure, one side of which is a wall. What will be the radius of the enclosure?*

Let XY be the wall and ALB the wire-netting. Let O be the centre of the circle and r miles the radius, while A square miles is the area of the enclosure, which must take the form of a segment of a circle, as in one or other of the two diagrams.

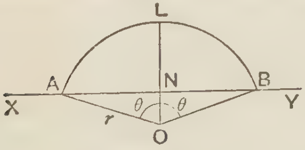


FIG. 127A.

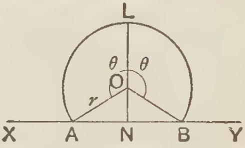


FIG. 127B.

Then
$$A=\text{sector OAB}\mp\triangle\text{OAB}$$
$$=\tfrac{1}{2}r^2\,.\,2\theta+\tfrac{1}{2}r^2\sin(2\pi-2\theta)$$
$$=r^2(\theta-\tfrac{1}{2}\sin 2\theta)\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

And $1 = \text{arc ALB} = 2r\theta$ (since there is 1 mile of netting).

Hence $r = \dfrac{1}{2\theta}$ ....(2)

whence, substituting from (2) into (1), we obtain

$$A = \frac{1}{4}\left(\frac{1}{\theta} - \frac{\sin 2\theta}{2\theta^2}\right).$$

$$\therefore\ 4A = \frac{1}{\theta} - \frac{\sin 2\theta}{2\theta^2}$$

Hence $$\frac{dA}{d\theta} = \frac{\cos^2\theta}{2\theta^3}(\tan\theta - \theta).$$

But, when A is a maximum, $$\frac{dA}{d\theta} = 0.$$

$$\therefore\ \frac{\cos^2}{\theta^3}\{\tan\theta - \theta\} = 0.$$

Now, a rough graph shows that there is no solution to $\tan\theta - \theta = 0$ between $\theta = 0$ and $\theta = \pi$, which are easily seen to be the two extreme possible values for θ.

$$\therefore\ \text{we must have } \cos\theta = 0,$$

$$i.e.\ \theta = 90^\circ.$$

Consequently the wire-netting forms a semicircle.

Hence $$r = \frac{1}{2 \times \frac{\pi}{2}} = \frac{1}{\pi} \text{ of a mile}$$

$$= 560 \text{ yards.}$$

6. Use of parameters.

A curve is usually given as an equation in x and y, *e.g.* $y = 3x + 4$, $x^2 + y^2 = 4$, $y^2 = 12x$.

In all branches of mathematical work it is, however, often convenient to express x and y in terms of a single "parameter." We shall give some examples which will show exactly what is meant.

Thus, in "parabolic motion" in Mechanics one expresses the coordinates of the moving body in terms of the time, which in that case is the "parameter."

Again, in dealing with the geometry of the ellipse, we express x and y as functions of the eccentric angle as the "parameter."

Example 1. If $\left\{\begin{array}{l} x=3t-2 \\ y=5t+4 \end{array}\right\}$ *let us assign values to the parameter* t, *calculate the corresponding values of* x *and* y, *and thereafter plot* y *against* x. *What sort of curve is obtained?*

We first construct the following table of corresponding values of t, x, and y:—

t	-5	-4	-3	-2	-1	0	1	2	3	4	5
x	17	-14	-11	-8	-5	-2	1	4	7	10	13
y	-21	-16	-11	-6	-1	4	9	14	19	24	29

Plotting, we obtain the following graph, which is plainly a straight line:—

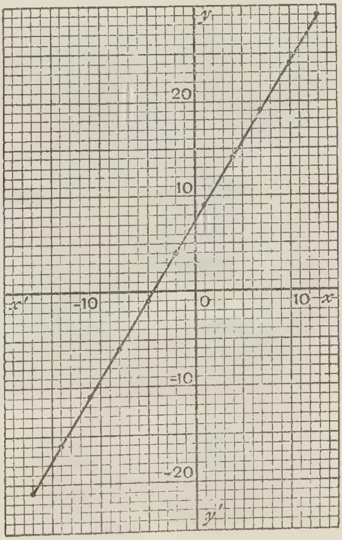


FIG. 128.

Example 2. *Plot y against x, where $x=0{\cdot}1t^2$ and $y=t$, for values of the parameter t ranging between $t=0$ and $t=20$. Draw the tangents at a point P on the curve cutting the x-axis at T. Draw PN perpendicular to the x-axis. If the curve cuts the x-axis in A, measure TA and AN. Do this for 5 points P on the curve. What geometrical property of the curve do you infer?*

We make the following tabulation :—

t	0	± 5	± 10	$\pm 10\sqrt{2}=14{\cdot}1$	$\pm 10\sqrt{3}=17{\cdot}3$	± 20
x	0	2·5	10	20	30	40
y	0	± 5	± 10	$\pm 14{\cdot}1$	$\pm 17{\cdot}3$	± 20

Plotting y against x, we obtain the following graph, which is a parabola :—

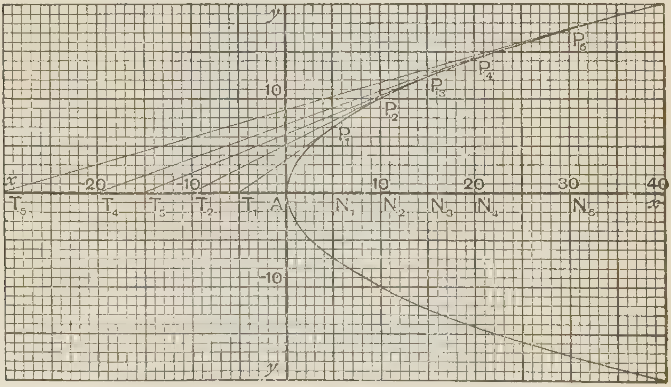


FIG. 129.

The graph is plainly symmetrical about the axis of x, and on drawing tangents at 5 different points we get in every case AN = AT.

Example 3. *If $x=8\cos\phi$ and $y=5\sin\phi$, assign values to the parameter ϕ from 0° to 360°, and obtain the corresponding values for x and y. Plot y against x on squared paper. At the point $\phi=47°$, draw the* *tangent to the curve as accurately as you can, and "read off" the intercepts that it makes on the axes.*

From the Tables we make the following tabulation, and then plot y against x:—

ϕ	20°	40°	60°	80°	90°	100°	120°	140°	160°	180°	0°	270°	360°
x	7·52	6·13	4	1·39	0	– 1·39	– 4	– 6·13	– 7·52	– 8	8	0	8
y	1·71	3·21	4·33	4·92	5	4·92	4·33	3·21	1·71	0	0	– 5	0

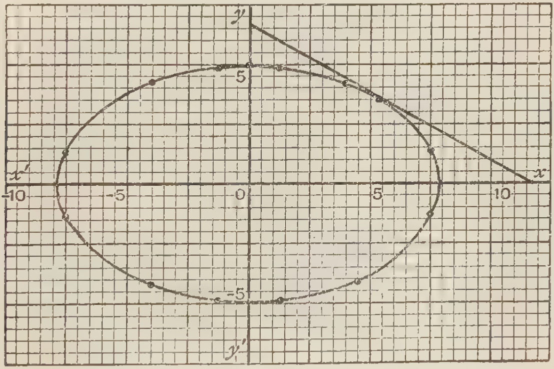


FIG. 130.

The curve is an ellipse and is plainly symmetrical about both axes.

Plotting, the point $x = 8 \cos 47^\circ = 5{\cdot}46$, $y = 5 \sin 47^\circ = 3{\cdot}66$, and drawing the tangent there we read off for intercepts on the axes made by the tangent 11·75 and 6·5.

7. To find the gradient of the tangent to a curve when x and y are given as functions of a parameter.

The gradient of the tangent is $\dfrac{dy}{dx}$, which may be written:

$$\frac{dy}{dx} \equiv \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}. \quad \text{(See Chap. IV. § 1.)}$$

We shall illustrate by some examples.

Example 1. *To find the gradient and inclination to the axis of x of the straight line in Example 1 of last article.*

Since and
$$\left.\begin{aligned} x&=3t-2 \\ y&=5t+4 \end{aligned}\right\}$$

$$\therefore \frac{dx}{dt}=3$$

and
$$\frac{dy}{dt}=5.$$

$$\therefore \frac{dy}{dx}\equiv\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{5}{3}=1{\cdot}6667.$$

Hence the "gradient" of the line is 1·6667, and its "inclination" to the axis of x is 59° 2′.

Example 2. *To find the gradient of the tangent at the point t in Example 2 of last article.*

We have
$$x=0{\cdot}1t^2,$$
$$y=t.$$

$$\therefore \frac{dx}{dt}=0{\cdot}2t,$$

$$\frac{dy}{dt}=1.$$

$$\therefore \frac{dy}{dx}\equiv\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{1}{0{\cdot}2t}=\frac{5}{t}.$$

In particular, when $t=3$, the slope ψ of the tangent is given by

$$\tan\psi\equiv\frac{dy}{dx}=\frac{5}{3}=1{\cdot}6667.$$

$$\therefore \psi=59^\circ\ 2'.$$

Example 3. *To find the gradient of the tangent at the point ϕ in Example 3 of last article.*

We have
$$\frac{dy}{dx}\equiv\frac{\frac{dy}{d\phi}}{\frac{dx}{d\phi}}=\frac{5\cos\phi}{-8\sin\phi}=-\frac{5}{8}\cot\phi.$$

Example 4. *A body is projected from the origin O. After t seconds the coordinates (x, y) of the position of the body are given by the parametric equations :*

$$x = 98t,$$
$$y = 175t - 16t^2.$$

Find after what time the direction of the body's motion will be making 45° with the horizontal.

As before,
$$\frac{dy}{dx} \equiv \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{175 - 32t}{98} \quad \text{.............................(1)}$$

But when the direction of the body's motion is making 45° with the horizon,

$$\frac{dy}{dx} \equiv \tan 45^\circ = 1 \quad \text{.............................(2)}$$

Hence by (1) and (2)
$$1 = \frac{175 - 32t}{98},$$

i.e. $t = 2{\cdot}4$,

i.e. after 2·4 secs.

Example 5. *To prove that the portion of the tangent to a rectangular hyperbola intercepted by the asymptotes is bisected at the point of contact.*

Let the parametric equations to the rectangular hyperbola $xy = c^2$ be

$$\left.\begin{aligned} x &= ct \\ y &= ct^{-1} \end{aligned}\right\}.$$

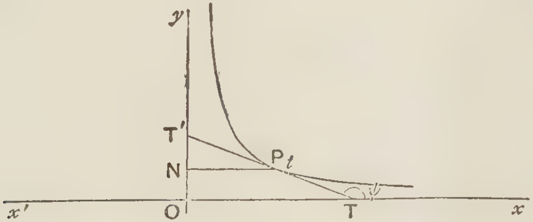


FIG. 131.

Then the equation to the line drawn from the point t inclined at angle ψ to the axis of x is, as usual,

$$\frac{y - ct^{-1}}{x - ct} = \tan \psi$$

$$=\frac{dy}{dx}$$

$$=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$=\frac{-ct^{-2}}{c}$$

$$=-t^{-2}$$

$$i.e.\ (y-ct^{-1})=-t^{-2}(x-ct)\ldots\ldots(1)$$

The line (1) cuts $x'x$, where (putting $y=0$ in (1))

$$x=2ct.$$

$$\therefore\ \mathrm{OT}=2ct\ \ldots\ldots(2)$$

Also $$\mathrm{PN}=ct\ \ldots\ldots(3)$$

$$\therefore\ \mathrm{OT}=2\mathrm{PN}$$

by (2) and (3), and TT′ is bisected at P.

8. To integrate differential equations of the type

$$\frac{d^n y}{dx^n}=f(x).$$

We integrate both sides directly with respect to x.

Example 1. *Find y in terms of x by means of the differential equation* $\frac{d^3y}{dx^3}=4x$, *where* $y=1$, $\frac{dy}{dx}=-2$, $\frac{d^2y}{dx^2}=3$ *when* $x=0$.

We have $\frac{d^3y}{dx^3}=4x$. Hence $\frac{d^2y}{dx^2}=2x^2+\mathrm{C}$.

$$\therefore\ \frac{d^2y}{dx^2}=2x^2+3 \text{ since } \mathrm{C}=3 \quad \because\ \frac{d^2y}{dx^2}=3 \quad \text{when } x=0.$$

$$\therefore\ \frac{dy}{dx}=\frac{2}{3}x^3+3x-2 \quad \because\ \frac{dy}{dx}=-2 \quad \text{when } x=0.$$

$$\therefore\ y=\frac{1}{6}x^4+\frac{3}{2}x^2-2x+1 \quad \because\ y=1 \quad \text{when } x=0.$$

Example 2. *A flexible ruler 2 ft. long is fixed horizontally into a wall, and a light weight is hung on at the free end. The differential* *equation determining the amount of deflexion y inches at a distance x inches from the wall is given by* $\dfrac{d^2y}{dx^2}=0\cdot0004(24-x)$. *If the deflexion of the free end be 2 inches, find y in terms of x.*

We have

$$\frac{d^2y}{dx^2}=0\cdot0004(24-x).$$

$$\therefore \frac{dy}{dx}=0\cdot0004(24x-\tfrac{1}{2}x^2)+\mathrm{K},$$

where K is some constant quantity which we must find.

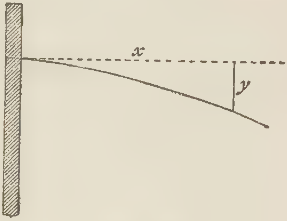


FIG. 132.

$$\therefore\ y=0\cdot0004(12x^2-\tfrac{1}{6}x^3)+\mathrm{K}x+\mathrm{K}',$$

where K′ is another constant quantity to be determined.

But $y=0$ when $x=0$.

$$\therefore\ \mathrm{K}'=0.$$

Also $y=2$ when $x=24$.

$$\therefore\ 2=0\cdot0004\times24^2(12-4)+24\mathrm{K}.$$

$$\therefore\ \mathrm{K}=\tfrac{1}{12}-0\cdot0004\times192$$

$$=0\cdot0833-0\cdot0768$$

$$=0\cdot0065.$$

$$\therefore\ y=0\cdot0004(12x^2-\tfrac{1}{6}x^3)+0\cdot0065x.$$

9. Radius of Curvature.

To find the radius of curvature at a given point of a curve.

Let P be a given point of the curve, and let Q be a point of the curve very near to P. Let the normals at P and Q (*i.e.* the lines perpendicular to the tangents at these points) meet in C. Let the tangents at P and Q be inclined to the axis of x at ψ and $\psi+\delta\psi$ respectively. Then, since PQ is a very short arc, we may regard it as very approximately the arc of a circle whose centre is at C. Let the radius of this circle be ρ.

Plainly the normals at P and Q (being perpendicular to the tangents) are inclined to one another at an angle equal to the angle between the tangents, *i.e.* $\delta\psi$.

Hence $$\text{PCQ} = \delta\psi.$$

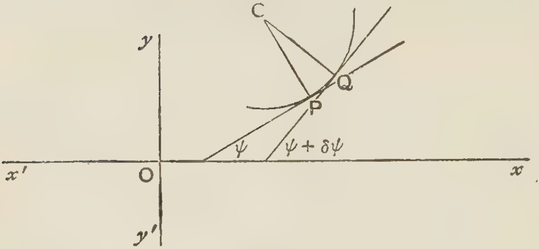


FIG. 133.

But very approximately the arc PQ = CP × PCQ (if PCQ be measured in radians).

Now, in the usual notation $$\text{PQ} \equiv \delta s$$
$$\text{CP} \equiv \rho$$
$$\text{PCQ} = \delta\psi.$$

Hence $$\delta s = \rho\, \delta\psi \text{ (very approx.)},$$

$$i.e.\ \frac{\delta s}{\delta\psi} = \rho \text{ (very approx.)}.$$

$$\therefore\ \frac{ds}{d\psi} = \rho \text{ in the limit.}$$

To express ρ in terms of x, y and their differentials.

We have $$\frac{dy}{dx} = \tan\psi.$$

$$\therefore\ \frac{d^2y}{dx^2} = \sec^2\psi\, \frac{d\psi}{dx}.$$

$$\therefore\ \frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \frac{d\psi}{ds} \cdot \frac{ds}{dx}.$$

$$\therefore\ \frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \rho^{-1} \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= \rho^{-1}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}.$$

$$\therefore\ \rho=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

Note.—If in the neighbourhood of P the curve is regarded as concave, then in the neighbourhood of Q the curve will be convex. The radius of curvature at P has the opposite sign to what it has at Q. At some point I between P and Q the curve is neither concave nor convex; in fact, it is very nearly a straight line. Hence, at I, ρ is infinite. Such a point as I where ρ is infinite is called a "*point of inflexion*," or simply a "*flex.*

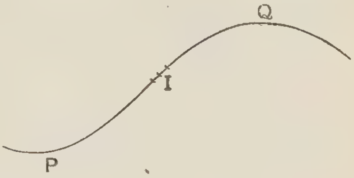


FIG. 134.

Since $\rho=\infty$, $\therefore\ \frac{d^2y}{dx^2}=0$.

Hence we have the following result:—

At a point of inflexion, $\frac{d^2y}{dx^2}$ *vanishes.*

Example 1. *To find the radius of curvature at the point* (1, 2) *on the curve* $y=3x^3-7x+6$.

We have $$\frac{dy}{dx}=9x^2-7$$

and $$\frac{d^2y}{dx^2}=18x.$$

Hence $$\rho=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$=\frac{\{1+(9x^2-7)^2\}^{\frac{3}{2}}}{18x}$$

$$=\frac{5\sqrt{5}}{18}\quad\text{when } x=1.$$

Example 2. *Sketch the curve* $y=x^3-3x^2-9x+5$ *and find its point of inflexion.*

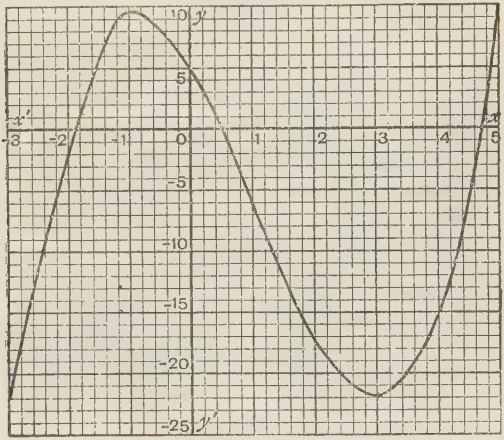


FIG. 135.

We have $$\frac{dy}{dx}=3(x^2-2x-3)$$

and $$\frac{d^2y}{dx^2}=6(x-1).$$

Hence $\frac{d^2y}{dx^2}=0$ when $x=1$, and consequently we have a "flex" at the point $(1, -6)$, as is evident from the sketch.

EXAMPLES XII.

1. Differentiate $x \operatorname{cosec} x$. $x^2 \sec x$, $x^{\frac{1}{2}} \tan x$, $(1+x) \cot x$, $x\sqrt{1+x}$, $x^2\sqrt{1+x}$, (x being measured in radians).

2. Find $\frac{dy}{dx}$ when $y=(7x-9)(x^2+15x-3)$.

3. Find $\frac{dy}{dx}$ when $x=1$, given that $y=(x^2+3)(2-4x-5x^2)$.

4. Find $\frac{dy}{dx}$ when $x=0$, given that $y=(x^3+3)(x+1)^4$.

5. Prove that $y = x\sqrt{1+x^2}$ satisfies the differential equation $y\dfrac{dy}{dx} = 2x^3 + x$.

6. If $y = \sin^2 x \cos^3 x$, find $\dfrac{dy}{dx}$ when $x = \dfrac{\pi}{2}$.

7. Differentiate $\sqrt{x^2-9} \times \sqrt{x^2-16}$, $x^3 \sin x$, $\sin^2 x \cos x$, $\sin^m x \cos^n x$, $\sec x \tan x$, $\operatorname{cosec} x \cot x$, $\sqrt{ax+b} \times \sqrt{cx+d}$, $7x^2 \cot^{\frac{1}{2}} x$, $(1-x^2)^{\frac{3}{2}}(1+x^2)^{\frac{5}{2}}$, $(1-x^3)(1+x^3)^{\frac{7}{3}}$, ($x$ being measured in radians).

8. Verify that $\dfrac{d \cos 2\theta}{d\theta} = -2 \sin 2\theta$, using the product formula $\cos 2\theta = (\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$.

9. A conical funnel is to be made to hold half a litre. What must be the semi-vertical angle of the cone and the radius of its circular base that the tin sheet used may be a minimum?

10. 1 square foot of tin sheet is to be formed into a conical funnel of maximum capacity. What will that capacity be if none of the tin sheet is cut to waste?

11. An angle of 80° is divided into two parts by a straight line. Find when the product of the sines of the two angles is a maximum.

12. An angle of 60° is divided into two parts by a straight line. Find when the product of the tangents of the two angles is a maximum.

13. At a fixed point O on a circle of 3 in. radius two perpendicular chords OA, OB are drawn. Find their lengths when rectangle OA.OB is a maximum.

(*Hint.*—AB is a diameter of the circle. Let angle OAB be θ radians. Express OA and OB in terms of the diameter and θ.)

14. ABCD is a square, and P is a point on CD. Find P when rectangle AP.BP is a minimum.

(*Hint.*—Let the side of the square be a cm. Let DP $= x$ cm. Express AP and BP each in terms of x and a.)

15. O is a fixed point on a circle of diameter d, and P and Q are points lying on the circle and symmetrical with respect to the diameter through O. Find the position of P so that triangle POQ is of maximum area.

16. Prove that the volume of a cylinder inscribed in a sphere of radius a cm. can be expressed by the formula $V = 2\pi a^3 \sin^2\theta \cos\theta$, and thence determine the cylinder of maximum volume which can be inscribed, where θ is the semi-angle of the cone subtended by the base of the cylinder at the centre.

17. Differentiate $\frac{\sin x}{x}$, $\frac{\cos x}{x^2}$, $\frac{\tan x}{x^3}$, $\frac{\sin x + \cos x}{x}$, $\frac{\sin x}{1+x}$, $\frac{\operatorname{cosec} x}{1+x}$, $\frac{\cot x}{1+x^2}$, $\frac{x}{1+x}$, $\frac{x^2}{1+x^2}$, (the angles being measured in radians).

18. Differentiate $\frac{\sec x}{x}$ as a quotient and $x^{-1} \sec x$ as a product, and verify that the same result will be obtained.

19. If $y = \frac{x}{1+x^2}$, find $\frac{dy}{dx}$ when $x = 1$.

20. If $y = \frac{\cos x}{1+\sin x}$, find $\frac{dy}{dx}$ when $x = \frac{\pi}{2}$.

21. If $y = \frac{\cot x}{1+\tan x}$, find $\frac{dy}{dx}$ when $x = \frac{\pi}{4}$.

22. If $y = \frac{2+x}{x^2+x+1}$, find $\frac{dy}{dx}$ when $x = 0$.

23. If $x = \frac{\sin \theta}{1+\sin \theta}$ and $y = \frac{\cos \theta}{1+\cos \theta}$, find $\frac{dy}{dx}$ in terms of θ, (θ being in radians).

24. If $x = \frac{\sin \theta}{\theta}$ and $y = \frac{\theta}{\sin \theta}$, find $\frac{dy}{dx}$ in terms of θ, (θ being in radians).

25. Differentiate $\frac{1+x}{1-x}$, $\frac{3-5x+7x^2}{4+2x-x^3}$, $\frac{\sqrt{1-x^2}}{\sqrt{1-x^3}}$, $\frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}+x}$.

26. Differentiate $\frac{\cos x}{\cos x + \sin x}$, $\frac{\sin x}{1+\tan x}$, $\frac{\sec x + \tan x}{\sec x - \tan x}$, $\frac{\operatorname{cosec} x - 1}{\cot x + 2}$, $\frac{\sin mx}{\cos nx}$, $\frac{\sin^m x}{\cos^n x}$, $\frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$, $\frac{\cot x - \cot^3 x}{1 + \cot^2 x}$, ($x$ being measured in radians).

27. O is a fixed point within the arms CA, CB of a right angle $\widehat{ACB}$, and a straight line AOB cuts the arms in A and B. Find the position of AOB when the area of the triangle ABC is as small as possible.

28. An electric bulb in a room on the ground floor is situated at a height of 8 ft. from the floor, and is distant 2 ft. 6 in. from a blind which is being lowered at the rate of 1 ft. per sec. Find the rate at which the shadow of the bottom of the blind advances on the ground outside the window when the bottom of the blind is 2 ft. above the floor.

29. If u inches be the distance of a candle-flame on one side of a lens, and v inches the distance of its image caught on a small screen on the other side, it is proved experimentally that $\frac{1}{u}+\frac{1}{v}=\frac{1}{3}$. Find when the candle-flame and image are closest together.

30. Solve last question in the general case when $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$.

31. The ceiling of a laboratory is 5 metres above the floor. On a wall 3 metres from the floor is suspended a pendulum 1 metre long. Underneath runs a bench on which, at a height of 1 metre from the floor and immediately beneath the bob of the pendulum in its lowest position, is an electric-light bulb. Compare the velocity of the shadow cast by the bob on the ceiling with the angular velocity of the pendulum when the pendulum makes an angle θ radians with the vertical.

32. A bath is such that to raise its temperature 1° C. requires the same heat as would raise 30 lb. of water 1° C. Initially its temperature is 15° C., and water at 90° C. is running into it from a geyser at the rate of 2 cub. ft. per min. What is the rate of rise of temperature of the water in the bath after 6 min., neglecting loss by radiation, etc.?

33. The other conditions being the same as in Question 32, what is the rate of rise of temperature after 2 min. when the loss of heat by radiation, etc., per min. would raise 450 lb. of water through 1°C.

34. Two straight roads meet at right angles, and two men, A and B, are walking on them, A toward the junction and B away from it. In the angle between the roads is situated a tree which is 40 ft. from A's road and 50 ft. from B's road, and initially B is 60 ft. from the junction, and A and B are in a line with the tree. Compare their speeds at this time that the tree may continue to hide one from the other.

35. A semicircular walk in a park has a radius of $\frac{1}{2}$ mile, and its two ends are joined by a straight walk passing through the centre of the semicircle. Two men, A and B, are situated, A at the centre of the semicircle and B on the semicircle, so that AB is perpendicular to the straight walk, and half way between them is a tree. B moves on the semicircle at 2 miles per hour. Find the speed of A on the straight walk 5 minutes later so that the tree may continue to hide one from the other.

36. Plot y against x where $x=2{\cdot}3-4{\cdot}5t$

$$y=1{\cdot}7+2{\cdot}1t.$$

Show that the graph is a straight line and "read off" its gradient. Verify by calculation.

37. Plot y against x where $x=3t-4$

$$y=0{\cdot}4t^2-t+2.$$

Draw the tangents at the points given by $t=0, 1, 3, -2, 0{\cdot}2, -1$, and "read off" their gradients. Verify by calculation.

38. Plot y against x where $x=7\cos\theta$

$$y=7\sin\theta.$$

Draw the tangents at the points given by $\theta=30°, 45°, 60°, 73°, 90°, 112°, 180°, 210°$, and "read off" their gradients. Verify by calculation.

39. Plot y against x where $x=7\cos\phi$

$$y=12\sin\phi.$$

Draw the tangents at the points given by $\phi=40°, 70°, 110°, 120°$, and "read off" their gradients. Verify by calculation.

40. Plot y against x where $x=5t$

$$y=\frac{7}{t}.$$

Draw the tangent at any point P on the curve given by t. If O be the origin, and the tangent at P meets the axes in L and M, calculate OL and OM in terms of t and find the law connecting them.

41. In Question 40, calculate the gradients of the tangents at the points given by $t=1, 3, 7, -4, -2$.

42. If $x=t^2+1$ and $y=t^3+1$, find $\frac{dy}{dx}$ in terms of t by differentiating, and hence calculate $\frac{dy}{dx}$ when $t=0, -1, 1, 2, -3, 2{\cdot}4, 5{\cdot}3$.

43. If $x=t+t^{-1}$ and $y=2t$, find t when $\frac{dy}{dx}=-1$.

44. If $x=3\cos\phi$ and $y=7\sin\phi$, find ϕ when $\frac{dy}{dx}=1{\cdot}5$, ϕ being measured in radians.

45. If $x=3\tan\phi$ and $y=4\cot\phi$, find $\frac{dy}{dx}$ in terms of ϕ, and hence calculate $\frac{dy}{dx}$ when $\phi=37°, 49°, 115°, 153°$.

46. If a particle be projected from a point with a velocity of v ft. per sec. in a direction making an angle of $\alpha°$ with the horizontal, it is proved in books on dynamics that the horizontal and vertical distances (x and y ft. respectively) from the point of projection after t seconds are given by

$$x = v \cos\alpha\, t$$
$$y = v \sin\alpha\, t - 16t^2.$$

Find $\frac{dy}{dx}$ in terms of t (*i.e.* find the direction in which the particle is moving after t sec.).

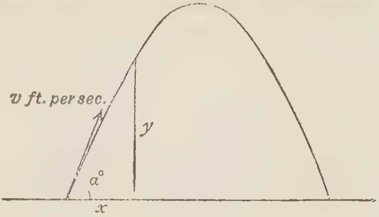


FIG. 136.

Find when the particle is at its highest point (*i.e.* find when the direction of flight is entirely horizontal).

Find when the particle is flying in a direction making 45° with the horizontal.

47. In Question 46, find the height of the particle when it is at its highest point of flight.

48. In Question 46, find when the particle reaches the ground again.

49. In Question 46, find the direction of the particle's motion just before it reaches the ground again.

50. Find the distance between the point of projection and the point where the particle reaches the ground again.

51. A and B are two points in the same horizontal line distant 10 ft. apart. A revolving drum, diameter 1 ft., moving at the rate of one revolution in 5 seconds, is situated at D, where BD = 8 ft. A

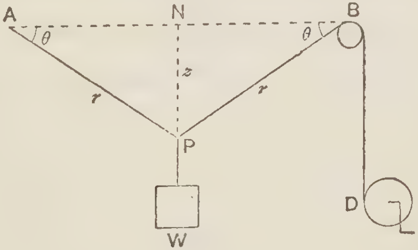


FIG. 137.

rope fixed at A and passing over a smooth pulley at B winds on to the revolving drum at D. P is a smooth pulley slung on the rope and supporting the weight W.

If AP $= r$ ft., express $\frac{dr}{dt}$ in terms of the rate of revolution of the drum.

Express r and z in terms of θ, and hence express the ratio of $\frac{dz}{dt}$ to $\frac{dr}{dt}$ in terms of θ.

Calculate $\frac{dz}{dt}$ when $\theta = 45°, 30°, 20°, 10°$.

52. In Question 51, find the angular velocity of AP when $\theta = 45°$, $30°$, $20°$, $10°$.

53. A certain gas which obeys Boyle's Law $pv = c^2$ has its pressure given by $c(t+1)$ dynes per sq. cm. t seconds after its initial condition. Find the rate of increase of the pressure.

Find also the rate of diminution of the volume initially and after 1, 2, and 3 seconds, the original volume being c cubic centimetres.

54. $x = a(\phi - \sin\phi)$
$y = a(1 - \cos\phi)$.

Take $a = 10$. Calculate the values of x and y for the following values of ϕ :—

$$0,\ \frac{\pi}{6},\ \frac{\pi}{4},\ \frac{\pi}{3}.$$

Plot points whose coordinates are these values of x and y on squared paper and draw a curve. [This curve is called the Cycloid.]
(Bd. of Educ.)

55. In the last question, find $\frac{dy}{dx}$ in terms of ϕ, and hence find a formula connecting ψ and ϕ where $\frac{dy}{dx} = \tan\psi$ as usual.

56 $\frac{d^2y}{dx^2} = 20x^3 - 42x + 8$. Find y in terms of x if $y = 4$ and $\frac{dy}{dx} = -11$ when $x = 1$.

57. $\frac{d^3y}{dx^3} = 120x^2$. Find y in terms of x if $y = 33$, $\frac{dy}{dx} = 119$, $\frac{d^2y}{dx^2} = 294$ when $x = 2$.

58. $\frac{d^2y}{dx^2} = 3\sin x$. Find y in terms of x if $y = 0$, $\frac{dy}{dx} = 0$ when $x = \frac{\pi}{2}$.

59. $\frac{d^2y}{dx^2} = 7\cos x - 5\sin x$. Find y in terms of x if $y = -7$, $\frac{dy}{dx} = 5$ when $x = 0$.

60. If a body moving down a certain inclined plane describes s feet in t seconds from rest, it is known that $\frac{d^2s}{dt^2} = 10$. Express s in terms of t.

61. Find the radius of curvature at the point (0, 4) on the curve $y = 4(1-x)^3$, and the coordinates of its point of inflexion.

62. Find the radius of curvature of the parabola $y^2 = 16x$ at its vertex.

CHAPTER XIII.

DIFFERENTIATION OF THE INVERSE TRIGONOMETRICAL FUNCTIONS.

1. To find the differential coefficient of $\sin^{-1} x$, in radian measure.

The subjoined diagram represents the graph of $y = \sin^{-1} x$, in radian measure.

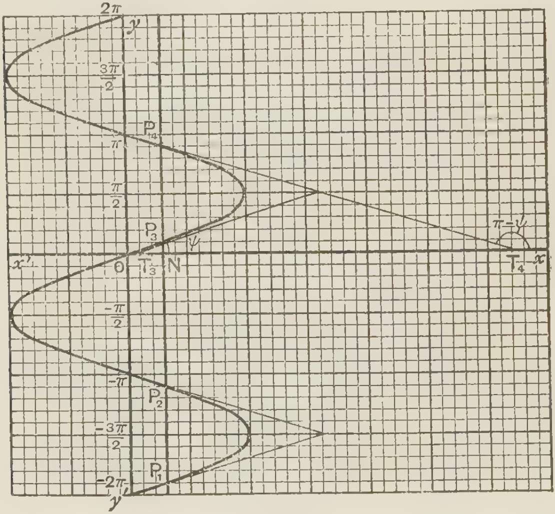


FIG. 138.

We note that, corresponding to a given value of x, viz. ON, there is an unlimited number of corresponding values of y, viz. NP_1, NP_2, NP_3, NP_4, etc. If we draw the tangents at P_1, P_2, P_3, P_4, etc., we see that the inclinations of these tangents to the axis of x have one or other of two values. If we denote the smaller of these values by ψ, it is plain from the diagram that the other is $\pi - \psi$..............................(1)

For if we consider a loop, it is plain that the tangents at P_3 and P_4 are equally inclined to the axis of y; whence the above property with regard to the axis of x at once follows.

To find $\frac{d\sin^{-1}x}{dx}$, we proceed thus :—

Let $y = \sin^{-1}x$ radians.

$$\therefore \quad x = \sin y \quad (y \text{ measured in radians}) \ldots\ldots (2)$$

$$\therefore \quad \frac{dx}{dy} = \cos y \quad \text{(by Chap. XI., Art. 3).}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\cos y} \quad \text{(by Chap. IV., Art. 1).}$$

$$\therefore \quad \frac{dy}{dx} = \pm \frac{1}{\sqrt{1-x^2}} \quad \{\text{by (2)}\} \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\therefore \quad \frac{d}{dx}(\sin^{-1}x) = \pm \frac{1}{\sqrt{1-x^2}}.$$

By the aid of the graph of $y = \sin^{-1}x$ (radians) we may render the question of the double sign precise, thus :—

Let $\sqrt{1-x^2}$ denote the positive value of the radical. Then we have the following chart :—

$\sin^{-1}x$ radians.	$\frac{d}{dx}\sin^{-1}x$.
. . . . .	. . . . .
$\frac{3\pi}{2}$ to 2π	$+\frac{1}{\sqrt{1-x^2}}$
π to $\frac{3\pi}{2}$	$-\frac{1}{\sqrt{1-x^2}}$
$\frac{\pi}{2}$ to π	$-\frac{1}{\sqrt{1-x^2}}$
0 to $\frac{\pi}{2}$	$+\frac{1}{\sqrt{1-x^2}}$
$-\frac{\pi}{2}$ to 0	$+\frac{1}{\sqrt{1-x^2}}$
π to $-\frac{\pi}{2}$	$-\frac{1}{\sqrt{1-x^2}}$
$-\frac{3\pi}{2}$ to $-\pi$	$-\frac{1}{\sqrt{1-x^2}}$
-2π to $-\frac{3\pi}{2}$	$+\frac{1}{\sqrt{1-x^2}}$
. . . . .	. . . . .

Example 1. *Calculate* $\frac{d\sin^{-1}x}{dx}$ *when* $x=0{\cdot}43$ *and radian measure is used,* $\sin^{-1}x$ *being an angle between* 0 *and* $\frac{\pi}{2}$.

Referring to the graph in this Article, we see that the differential coefficient is positive.

Hence
$$\frac{d\sin^{-1}x}{dx}=\frac{1}{\sqrt{1-(0{\cdot}43)^2}}$$
$$=1{\cdot}108.$$

Corollary. $\frac{d\sin^{-1}x}{dx}=-1{\cdot}108$ if $\sin^{-1}x$ lies between $\frac{\pi}{2}$ and π.

Example 2. *A kite is flying horizontally* 350 *yards above the ground, and string is being paid out at the rate of half a yard a second. How fast is the inclination of the string to the ground altering when* 800 *yards have been paid out?*

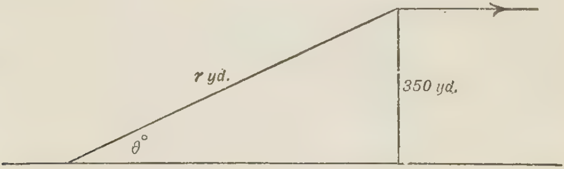


FIG. 139.

Let θ be the angle in degrees.

$$\therefore\ \sin\theta=\frac{350}{r}$$

$$\therefore\ \theta=\sin^{-1}\left(\frac{350}{r}\right).\ \dots\dots\dots\dots\dots\dots(1)$$

We wish to find $\frac{d\theta}{dt}$.

By (1)
$$\frac{d\theta}{dt}=\frac{d}{dt}\sin^{-1}\left(\frac{350}{r}\right)$$
$$=\frac{d\sin^{-1}\left(\frac{350}{r}\right)}{d\left(\frac{350}{r}\right)}\times\frac{d\left(\frac{350}{r}\right)}{dr}\times\frac{dr}{dt}$$

$$= \pm \frac{180}{\pi\sqrt{1-\left(\frac{350}{r}\right)^2}} \times \left(-\frac{350}{r^2}\right) \times \frac{dr}{dt}$$

$$= \pm \frac{180 \times 350}{\pi r \sqrt{r^2 - 350^2}} \times \frac{dr}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now $\frac{dr}{dt} = \frac{1}{2}$ yd. per sec. and $r = 800$ yds.

$$\therefore \frac{d\theta}{dt} = \pm \frac{180 \times 350}{1600\pi\sqrt{800^2 - 350^2}} \text{ degrees per sec.}$$

$$= \pm 0{\cdot}055 \text{ degrees per sec.}$$

If the kite be flying away from the man, θ is plainly decreasing and we take the negative sign.

If the kite be flying towards the man, θ is plainly increasing and we take the positive sign. In this latter case the string is being taken in.

2. Differentiation of $\cos^{-1}$ x measured in radians.

Consider $y = \cos^{-1} x$, where y is measured in radians.

Plainly ψ has two values here (corresponding to a given value of x), which are supplementary to one another.

Hence $\frac{dy}{dx}$ has two values, equal but of opposite signs. A figure or some other relevant circumstance will show which sign has to be taken in a given case.

Proceeding as in the case of $\sin^{-1} x$, we have :—

Let $$y = \cos^{-1} x.$$

$$\therefore \quad x = \cos y.$$

$$\therefore \quad \frac{dx}{dy} = -\sin y.$$

$$\therefore \quad \frac{dx}{dy} = \mp\sqrt{1 - x^2}.$$

$$\therefore \quad \frac{dy}{dx} = \mp \frac{1}{\sqrt{1 - x^2}}.$$

From the graph of $y = \cos^{-1} x$, we therefore see how the sign has to be taken, it being assumed that the positive value of $\sqrt{1 - x^2}$ is taken.

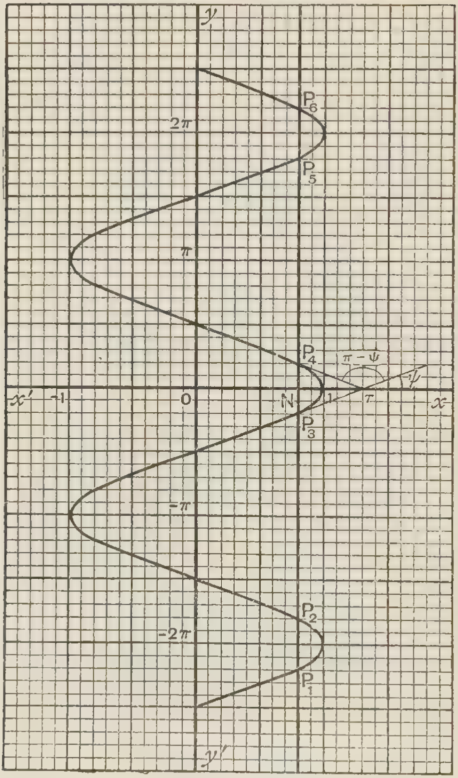


FIG. 140.

. . . .

From -2π to $-\pi$, $\dfrac{d\cos^{-1}x}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$,

From $-\pi$ to 0, $\dfrac{d\cos^{-1}x}{dx} = +\dfrac{1}{\sqrt{1-x^2}}$,

From 0 to $+\pi$, $\dfrac{d\cos^{-1}x}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$, From $+\pi$ to $+2\pi$, $\dfrac{d\cos^{-1}x}{dx}=+\dfrac{1}{\sqrt{1-x^2}}$,

and so on.

3. To differentiate $\tan^{-1} x$ in radian measure.

Consider the graph of $y=\tan^{-1}x$, radian measure being used.

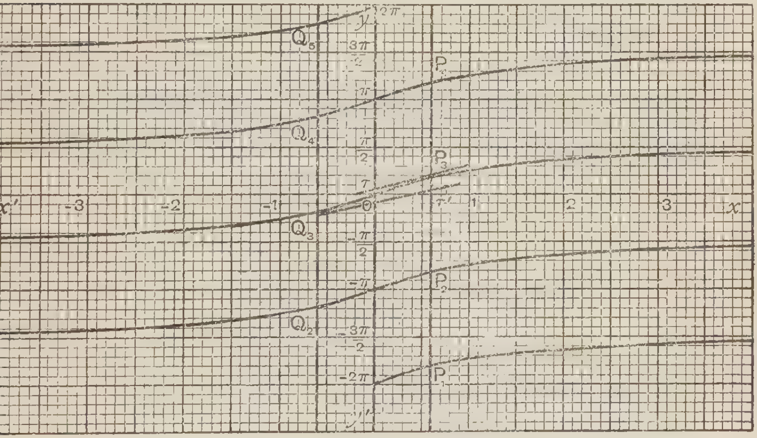


FIG. 141.

We see that the tangents at all the points P_1, P_2, P_3, P_4, . . . are parallel, and also those at Q_2, Q_3, Q_4, Q_5, . . . are parallel. Hence, if x be given, $\dfrac{dy}{dx}$ will have only *one* value corresponding. Moreover, ψ and ψ' being both acute angles, $\dfrac{dy}{dx}$ is always positive.

We have $$y=\tan^{-1}x.$$

$$\therefore\ x=\tan y.$$

$$\therefore\ \frac{dx}{dy}=\sec^2 y.$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{\sec^2 y}.$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{1+\tan^2 y}.$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{1+x^2}$$

(which verifies that, if x be given, there is only *one* corresponding value for $\left.\frac{dy}{dx}\right)$.

Example 1. *A monkey is ascending a pole 200 yds. away, and a boy is watching it with a toy telescope. Find the rate at which the telescope is being turned up when the monkey is 90 ft. up the pole and moving at $\frac{3}{4}$ ft. per sec.*

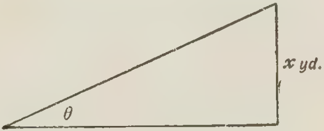


FIG. 142.

Let θ be measured in degrees.

$$\therefore\ \theta=\tan^{-1}\frac{x}{200}.$$

$$\therefore\ \frac{d\theta}{dt}=\frac{d}{dt}\tan^{-1}\frac{x}{200}$$

$$=\frac{d}{d\left(\frac{x}{200}\right)}\tan^{-1}\frac{x}{200}\times\frac{d\left(\frac{x}{200}\right)}{dt}\text{ (degrees per sec.)}$$

$$=\frac{180}{200\pi}\,\frac{1}{1+\frac{x^2}{40{,}000}}\,\frac{dx}{dt}\text{ (degrees per sec.)}\ldots\ldots\ldots\ldots\ldots(1$$

Now $\frac{dx}{dt}=\frac{3}{4}$ ft. per sec.

$=\frac{1}{4}$ yd. per sec.

$$\therefore \frac{d\theta}{dt} = \frac{0{\cdot}225}{\pi\left(1+\frac{9}{400}\right)} \text{ degrees per sec.}$$

$$= 0{\cdot}07 \text{ degrees per sec.}$$

Example 2. *At O is a small revolving mirror reflecting a spot of light on to a screen 30 cm. away. The mirror at O is revolving with a constant angular velocity of 0·5° per sec. Find the position of P when its velocity on the screen is 1 cm. per sec.*

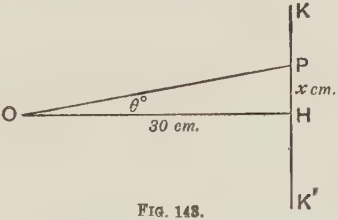


FIG. 148.

Let $\quad HP = x$ cm.

$$\therefore x = 30 \tan \theta^\circ.$$

$$\therefore \theta = \tan^{-1}\frac{x}{30}.$$

$$\therefore \frac{d\theta}{dt} = \frac{180}{\pi} \times \frac{30}{30^2+x^2}\frac{dx}{dt}.$$

But $\quad \frac{d\theta}{dt} = 1^\circ$ per sec. ($\because$ beam of light rotates twice as fast as mirror)

and $\quad \frac{dx}{dt} = 1$ cm. per sec.

$$\therefore x^2 + 900 = \frac{5400}{\pi}.$$

$$= 1719 \text{ (approx.)}.$$

$$\therefore x = 28{\cdot}6 \text{ cm.}$$

4. The following integrals are now obvious :—

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \text{ (radians)}, \left(0 < x < \frac{\pi}{2}\right)$$

$$= -\sin^{-1} x \text{ (radians)}, \left(\frac{\pi}{2} < x < \pi\right)$$

and so on.

$$\int \frac{dx}{1+x^2} = \tan^{-1} x \text{ (radians).}$$

Example 1. *Evaluate* $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$, *tracing first of all the graph of* $y = \frac{1}{\sqrt{1-x^2}}$.

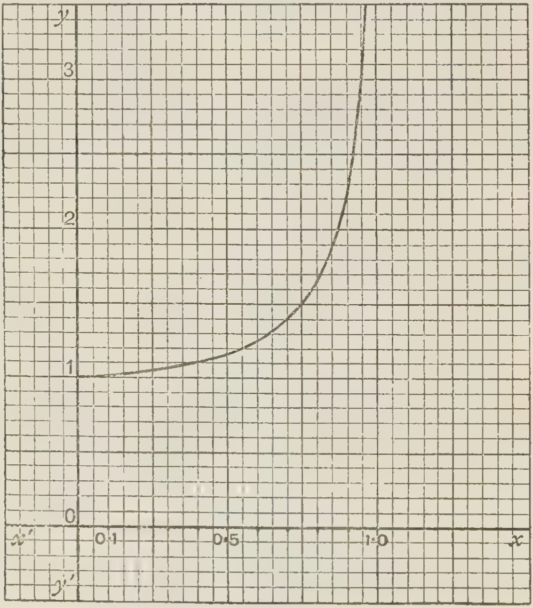


FIG. 144.

We have

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \Big[\sin^{-1} x\Big]_0^1$$

$$= \sin^{-1} 1 - \sin^{-1} 0$$

$$= \frac{\pi}{2}$$

$$= 1{\cdot}5708.$$

This represents the area enclosed between the graph of $y=\frac{1}{\sqrt{1-x^2}}$, the axes of x and y, and the ordinate corresponding to $x=1$.

Example 2. *Evaluate* $\int_0^{2\cdot3}\frac{dx}{1+x^2}$, *tracing first of all the graph of* $y=\frac{1}{1+x^2}$.

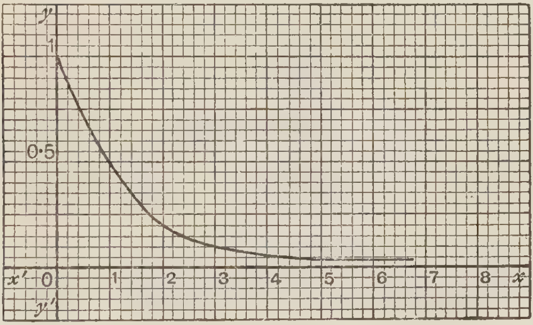


FIG. 145.

We have
$$\int_0^{2\cdot3}\frac{dx}{1+x^2}=\Big[\tan^{-1}x\Big]_0^{2\cdot3}$$
$$=\tan^{-1}2\cdot3-\tan^{-1}0$$
$$=\frac{\pi}{180}\times 66\cdot5 \quad \text{from the Tables}$$
$$=1\cdot161.$$

5. Vibrations.

Imagine a tuning-fork HO held in a vice at H, and imagine the free end P plucked aside to A and then left to vibrate. If the distance on either side of O through which the free end vibrates be small, we may regard the path of P, viz. A′OA, as very approximately a straight line.

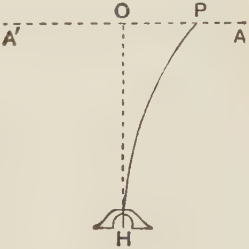


FIG. 146.

Let $OP=x$, $A'O=OA=a$.

It is an experimental result that, when the free end is at P, the elastic force tending to restore P to the position of equilibrium O is proportional to the distance x that P is from the position of equilibrium. But this force is proportional to the acceleration $\frac{d^2x}{dx^2}$. Hence $\frac{d^2x}{dt^2}$ is proportional to x. Since the force tending to bring back P to O is negative in direction, we must therefore take

$$\frac{\frac{d^2x}{dt^2}}{x} = -n^2,$$

where n^2 is a positive number, taken in the square form for convenience. Hence the differential equation for a vibratory motion is:

$$\frac{d^2x}{dt^2} + n^2x = 0.$$

To integrate this equation, we put $\frac{dx}{dt} \equiv v$.

$$\therefore \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dx}{dt} \cdot \frac{dv}{dx} = v\frac{dv}{dx} = \frac{1}{2}\frac{d(v^2)}{dx}.$$

Hence our differential equation becomes:

$$\frac{1}{2}\frac{dv^2}{dx} + n^2x = 0.$$

$$\therefore v^2 + n^2x^2 = \text{C}.$$

Now $v = 0$ when the free end of the tuning-fork is at A, *i.e.* when $x = a$. Hence $\text{C} = n^2a^2$.

$$\therefore v^2 = n^2(a^2 - x^2).$$

$$\therefore \frac{dx}{dt} = n\sqrt{a^2 - x^2}.$$

$$\therefore n\,dt = \frac{dx}{\sqrt{a^2 - x^2}}, \text{ (as it is conveniently written).}$$

$$\therefore \int n\,dt = \int \frac{dx}{\sqrt{a^2 - x^2}}.$$

$$\therefore\ nt + \mathrm{A} = \sin^{-1}\frac{x}{a}.$$

$$\therefore\ x = a\sin(nt + \mathrm{A}).$$

Now, when $t = 0$, the free end is plucked aside to A, *i.e.* $x = a$. Hence $\mathrm{A} = \frac{\pi}{2}$.

$$\therefore\ x = a\sin\left(nt + \frac{\pi}{2}\right).$$

$$\therefore\ x = a\cos nt.$$

Corollary I. *The time of a complete vibration is* $\frac{2\pi}{n}$.

For $$x = a\cos nt = a\cos(nt + 2\pi) = a\cos n\left(t + \frac{2\pi}{n}\right).$$

Hence, whatever position x is occupied by P at time t, that same position x will be occupied by P again at time $t + \frac{2\pi}{n}$. Hence P traverses the same position travelling in the *same* direction at intervals of time $\frac{2\pi}{n}$.

Corollary II. *If the time of a complete vibration is* $\frac{2\pi}{n}$ *sec., then the number of vibrations per second is* $\frac{n}{2\pi}$.

Note I. a is called the "*amplitude*" of the vibration, *i.e.* half the complete distance through which a disturbed body vibrates. In the case of a plucked string, a determines the loudness of the note produced.

Note II. $\frac{2\pi}{n}$ is called the "*period*" of the vibration. In the case of a plucked string it would determine the pitch of the note, A, B, C, etc., but not its loudness.

EXAMPLES XIII.

1. Differentiate $\sin^{-1}x+\cos^{-1}x$, $3\sin^{-1}x+5\cos^{-1}x$, $4\sin^{-1}x-7\cos^{-1}x$, $\sin^{-1}\frac{x}{3}$, $\sin^{-1}(1+x)$, $\sin^{-1}(1-x^2)$, $\cos^{-1}x^2$, $\cos^{-1}\frac{x}{7}$, $\cos^{-1}5x$, each angle being measured in radians.

2. Differentiate $x\sin^{-1}x$, $x^2\cos^{-1}x$, $(1+x)\tan^{-1}x$, $\frac{\sin^{-1}x}{x}$, $\frac{\cos^{-1}x}{x^2}$, $\frac{\tan^{-1}x}{1+x}$, each angle being measured in radians.

3. Sketch as accurately as you can the graph of $y=\sin^{-1}x$, and calculate the inclination of the tangent to the curve at the origin, y being measured in radians.

4. Sketch as accurately as you can the graph of $y=\tan^{-1}x$, and calculate the inclination of the tangent to the curve at the origin, y being measured in radians.

5. In Question 3, find the inclination of the tangent to the curve to the axis of x when $x=0{\cdot}3$.

6. In Question 4, find the inclination of the tangent to the curve to the axis of x when $x=0{\cdot}7$.

7. Calculate $\frac{d\sin^{-1}x}{dx}$ when $x=0{\cdot}81$, the angle being measured in radians.

8. Calculate $\frac{d\sin^{-1}x}{dx}$ when $x=0{\cdot}69$, the angle being measured in degrees.

9. Calculate $\frac{d\cos^{-1}x}{dx}$ when $x=0{\cdot}45$, the angle being measured in radians.

10. Calculate $\frac{d\cos^{-1}x}{dx}$ when $x=0{\cdot}92$, the angle being measured in degrees.

11. Calculate $\frac{d\tan^{-1}x}{dx}$ when $x=0{\cdot}87$, the angle being measured in radians.

12. Calculate $\frac{d\tan^{-1}x}{dx}$ when $x=2{\cdot}5$, the angle being measured in degrees.

13. Using the result of Chap. XI. Art. 7, find $\frac{dy}{dx}$ when $y = \operatorname{cosec}^{-1} x$.

14. Using the result of Chap. XI. Art. 8, find $\frac{dy}{dx}$ when $y = \sec^{-1} x$.

15. Using the result of Chap. XI. Art. 9, find $\frac{dy}{dx}$ when $y = \cot^{-1} x$.

16. P is a point moving along a straight line at the rate of v cm. per sec. O is a point situated a cm. from the above line. Find the angular velocity of OP in radians per sec. when P is distant x cm. from the foot of the perpendicular dropped from O on the given line.

$\left[\textit{Hint.}\text{—}\theta = \tan^{-1} \frac{x}{a}. \right]$

17. A point P moves round the circumference of a circle of radius a cm. PN is the perpendicular dropped from P on a fixed diameter. If O be the centre, express the angular velocity of OP in terms of a, the distance x cm. of N from O and the velocity v cm. per sec. of N.

18. A 40-ft. ladder is being raised from the ground, and when the moving end is 6 ft. from the ground it is rising vertically at the rate of 1·25 ft. per sec. What is the angular velocity with which the ladder revolves at this instant about its stationary end?

19. A galvanometer mirror is 1 metre distant from the scale, and the spot of light is moving on the scale with a velocity of 15 cm. per sec. when it is deflected 17 cm. What is the angular velocity of the beam of light at that instant, and what is the angular velocity of rotation of the mirror at the same instant?

20. The jib of a crane is 10 ft. in length, and it raises a body vertically at the rate of 3 in. per sec. What is the angular velocity of rotation of the jib when it is horizontal, and when it makes angles of 20°, 40°, 60° with the horizontal?

21. In a hundred yards race the winner passes the judge at 28 ft. per sec. and is distant 12 ft. from him. What is the angular velocity of the line joining the judge and the winner when the latter passes the tape?

22. An airship flying at an altitude of 10,000 ft. at 40 m.p.h. passes directly over a camouflaged anti-aircraft gun. What is the angular velocity with which the gun must be rotated in a vertical plane when the airship is distant 12,000 ft., 15,000 ft., 18,000 ft., so that the target may be kept on the sights?

23. An aeroplane flies at 100 m.p.h. at an altitude of 8000 ft. in a vertical plane. An anti-aircraft gun distant 9000 ft. from this

plane opens fire on it. With what angular velocity, in the plane containing the line of travel of the aeroplane and the gun, must the latter be rotated so as to keep the target on the sights when the aeroplane is at the shortest distance from the gun?

24. In the preceding question, what are the angular velocities of rotation of the gun when the target is distant 14,000 ft., 17,000 ft., 20,000 ft.?

25. An angler is fishing from the top of a bridge, the end of his rod being 20 ft. above the level of the stream. He has a "bite," and the fish drags his float downstream so that the line runs out at the rate of 24 ft. per min. What are the angular velocities of rotation of the line from the rod to the float when the fish is 10 ft., 20 ft., 30 ft., 40 ft. downstream from the bridge?

26. Evaluate $\int_{0\cdot5}^{0\cdot7}\frac{dx}{\sqrt{1-x^2}}$, $\int_{0\cdot3}^{0\cdot4}\frac{dx}{\sqrt{1-x^2}}$, $\int_{0\cdot19}^{0\cdot71}\frac{dx}{\sqrt{1-x^2}}$, $\int_{0\cdot62}^{0\cdot92}\frac{dx}{\sqrt{1-x^2}}$.

27. Evaluate $\int_2^3\frac{dx}{\sqrt{9-x^2}}$, $\int_0^4\frac{dx}{\sqrt{16-x^2}}$, $\int_{2\cdot3}^{4\cdot7}\frac{dx}{\sqrt{25-x^2}}$, $\int_{1\cdot31}^{5\cdot31}\frac{dx}{\sqrt{49-x^2}}$.

28. Evaluate $\int_0^{0\cdot3}\frac{dx}{\sqrt{1-4x^2}}$, $\int_0^{0\cdot21}\frac{dx}{\sqrt{1-9x^2}}$, $\int_0^{0\cdot11}\frac{dx}{\sqrt{1-25x^2}}$, $\int_0^{0\cdot07}\frac{dx}{\sqrt{1-81x^2}}$.

29. Evaluate $\int_1^5\frac{dx}{x\sqrt{x^2-1}}$, $\int_2^7\frac{dx}{x\sqrt{x^2-1}}$, $\int_{3\cdot5}^{9\cdot26}\frac{dx}{x\sqrt{x^2-1}}$, $\int_{1\cdot73}^{6\cdot26}\frac{dx}{x\sqrt{x^2-1}}$.

30. Evaluate $\int_1^2\frac{dx}{1+x^2}$, $\int_3^7\frac{dx}{1+x^2}$, $\int_{0\cdot5}^{1\cdot7}\frac{dx}{1+x^2}$, $\int_{1\cdot36}^{2\cdot78}\frac{dx}{1+x^2}$.

31. Evaluate $\int_0^1\frac{dx}{1+9x^2}$, $\int_{0\cdot4}^{1\cdot17}\frac{dx}{1+25x^2}$, $\int_{0\cdot25}^{0\cdot75}\frac{dx}{1+16x^2}$, $\int_{0\cdot1}^{0\cdot81}\frac{dx}{1+49x^2}$.

32. Evaluate $\int_0^5\frac{dx}{4+x^2}$, $\int_1^3\frac{dx}{9+x^2}$, $\int_{5\cdot55}^{10\cdot25}\frac{dx}{25+x^2}$, $\int_{24}^{57}\frac{dx}{36+x^2}$.

33. In treatises on Mechanics it is shown that if a small heavy bob be attached to a thin string of length l cm., the other end of which is fixed, and the bob be caused to vibrate, then if the string makes an angle θ radians with the vertical after t sec. we have $l\frac{d^2\theta}{dt^2}=-g\sin\theta$, where g is the acceleration due to gravity. Also it can be seen by reference to the Tables that $\sin\theta$ is very approximately equal to θ if

the angle does not exceed 4°. Thus $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$ for small vibrations of amplitude α radians.

(i) Express θ in terms of t and α, t being measured from the moment when the bob is in its lowest position.

(ii) Find the time of a complete vibration, and show that it is independent of α.

34. If a heavy rod of mass M grams and of length l cm. be pivoted at one end and be allowed to vibrate through a small angle α radians on each side of the vertical, then if its Moment of Inertia about the axis of suspension be I C.G.S. units, it can be proved that $I\frac{d^2\theta}{dt^2} = -Mgl\theta$, where θ radians is the angle made with the vertical t seconds after passing the vertical position.

(i) Express θ in terms of t and α.

(ii) Find the time of a complete vibration, and show that it is independent of α.

35. A magnet whose magnetic moment is M C.G.S. units and whose Moment of Inertia about an axis through its centre and perpendicular to its length is I C.G.S. units is freely suspended in the Earth's magnetic field, whose strength is H gauss. It is displaced from the North-and-South position through α radians, a small angle, and left to vibrate. Then it can be shown that, if after t sec. from being in the North-and-South position it makes an angle θ radians with that position, $I\frac{d^2\theta}{dt^2} = -MH\theta$.

(i) Express θ in terms of t and α.

(ii) Find the time of a complete vibration, and show that it is independent of α.

CHAPTER XIV.

APPROXIMATIONS AND ERRORS.

1. In the present chapter we shall deal with the method of obtaining a differential coefficient approximately under circumstances when the actual value cannot be found. We shall also show how to deal with calculations based on numerical data taken down in experiments performed in the laboratory, where these data are liable to "errors of observation."

The method of treatment in both cases is practically the same, and is based on the following principle, which we shall regard as graphically evident:

Connection between $\frac{dy}{dx}$ *and* $\frac{\delta y}{\delta x}$.

Let P and Q be two points near to one another on the graph under consideration, and let PL be the tangent at P.

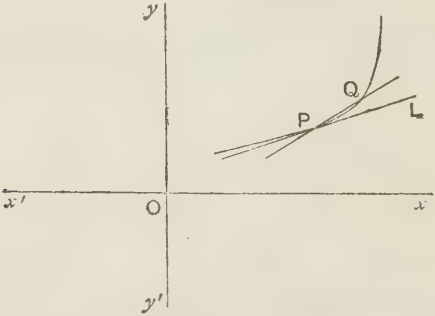


FIG. 147.

Then plainly the gradient of the chord PQ is very nearly equal to the gradient of the tangent at P.

Now the gradient of the chord PQ is $\frac{\delta y}{\delta x}$ (by Chap. II. Art. 3).

Also the gradient of the tangent at P is $\frac{dy}{dx}$.

Hence $$\frac{\delta y}{\delta x}=\frac{dy}{dx} \text{ (very approximately).}$$

2. Application of the above principle to approximate evaluations of a rate of change.

Very often the data will not allow us to calculate accurately $\frac{dy}{dx}$, and we are in that case compelled to compute $\frac{\delta y}{\delta x}$ as accurately as the data will allow. A few examples will show the method of procedure.

Example 1. *To find from a table of Natural Sines,* $\frac{dy}{dx}$ *when* $x=37°$, *where* $y=\sin x$.

We have from the tables

$$y=0{\cdot}6018 \text{ (i.e. } \sin 37°),$$
$$y+\delta y=0{\cdot}6032 \text{ (i.e. } \sin 37°\ 6'),$$
$$\therefore\ \delta y=0{\cdot}0014.$$

Also $$\delta x=0{\cdot}1 \text{ degree.}$$
$$\therefore\ \frac{\delta y}{\delta x}=0{\cdot}014.$$
$$\therefore\ \frac{dy}{dx}=0{\cdot}014 \text{ (very approximately) when } x=37°.$$

Verification.

$$y=\sin x°.$$
$$\therefore\ \frac{dy}{dx}=0{\cdot}01745\cos x° \text{ (by Chap. XI. Art. 3, Corollary II.).}$$
$$=0{\cdot}01745\times 0{\cdot}7986 \text{ (when } x°=37°).$$
$$=0{\cdot}0139 \text{ (correct to the fourth place of decimals).}$$

Example 2. *To find from a table of logarithms* $\frac{dy}{dx}$ *when* $x=17$, *where* $y=\log_{10}x$.

We have from the tables

$$y=1{\cdot}2304 \text{ (i.e. } \log_{10} 17),$$

$$y+\delta y=1{\cdot}2330 \text{ (i.e. } \log_{10} 17{\cdot}1)$$

$$\therefore\ \delta y=0{\cdot}0026.$$

Also $$\delta x=17{\cdot}1-17=0{\cdot}1.$$

$$\therefore\ \frac{\delta y}{\delta x}=0{\cdot}026.$$

$$\therefore\ \frac{dy}{dx}=0{\cdot}026 \text{ (very approximately) when } x=17.$$

Example 3. *In Whitaker's Almanack we find that the number of people in England and Wales in* 1801 *was* 8,892,536, *and in* 1811, 10,164,256. *To find the mean rate of increase in the population in these* 10 *years.*

The population has increased by 1,271,720 in 10 years. Hence the mean rate of increase during the period 1801–1811 is 127,172 per annum.

If we state this in our more symbolical language, P denoting the population and T the time in years, we have

$$\delta P=1{,}271{,}720,$$

$$\delta T=10.$$

$$\therefore\ \frac{\delta P}{\delta T}=127{,}172 \text{ persons per annum}$$

$$=127{,}000 \text{ (in round numbers).}$$

Example 4. *A sliding piece is at a distance of* 1·73 *cm. from a marked point in its path at a time* 0·1 *sec. It is at a distance of* 1·85 *cm. at a time* 0·2 *sec. Find the mean speed in the above interval of time.*

Let the sliding piece be x cm. from the marked point after t seconds.

Then $$\delta x=1{\cdot}85-1{\cdot}73=0{\cdot}12 \text{ centimetres,}$$

$$\delta t=0{\cdot}2\ -0{\cdot}1\ =0{\cdot}1 \text{ seconds.}$$

$$\therefore \text{ Mean speed} = \frac{\delta x}{\delta t}$$

$$= \frac{0{\cdot}12}{0{\cdot}1} = 1{\cdot}2 \text{ cm. per sec.}$$

$$\therefore \frac{dx}{dt} = 1{\cdot}2 \text{ cm. per sec. (very approx.) at the time and position under consideration.}$$

3. Errors of observation.

"Errors of observation" are of continual occurrence in all practical work in mechanics, electricity, geometrical measurements, etc., and it is therefore essential to study the mathematical methods of making allowance for such "errors of observation" in calculations based on experimental data taken down in the laboratory and involving such experimental errors.

We give some concrete examples of the occurrence in practical work of "errors of observation."

Example 1. *An ordinary school ruler can measure a line correct to* two *places of decimals. Hence the calculation of the area of the square described on this line can only be approximate. We require to find, therefore, between what limits of error we can rely on the calculation of the area of the square as being correct.*

Example 2. *An ordinary school protractor can measure an angle correct to* one *place of decimals. Having found this angle correct to one place of decimals, we look up its sine, cosine, etc., in a book of mathematical tables. We have therefore to find between what limits of error we can rely on this value of the sine, cosine, etc., seeing that the angle itself can be relied on only within certain prescribed limits.*

Example 3. *A certain instrument commits a certain systematic known error (e.g. a ruler may be under a foot by $\frac{1}{8}$ in.). We have to estimate the error made in calculations based thereon.*

4. Formulæ and experimental data.

For practical purposes we are concerned with the following principle.

A certain formula connecting two unknowns, x and y, has been obtained from theoretical considerations.

In using this formula in the laboratory, we wish to calculate y being given a value for x, this latter value for x having been obtained from experiments and hence being only approximately correct. It is necessary for us to find the error in the corresponding calculated value of y.

We may state the matter symbolically thus:—

(1) There is a formula $y=f(x)$ obtained from theoretical considerations.

(2) x is obtained approximately by means of experiments.

(3) We have to estimate the error in y which is calculated by means of the above formula, consequent on the error in x.

We use the formula of Art. 1 of this chapter, viz.:

$$\frac{\delta y}{\delta x}=\frac{dy}{dx} \text{ (very approx.)}$$

in the form
$$\delta y=\frac{dy}{dx}.\,\delta x.$$

N.B.—Since δy is the error in y, the percentage error is $\frac{100\delta y}{y}\%$.

We will give some illustrative examples of the application of this principle.

Example 1. *The formula giving the area of a square on a line x cm. long is $A=x^2$. x is measured and found to be 4·7 cm. long, but this measurement is 4% too short. Find the consequent error in* A.

$$\because\ A=x^2$$

$$\therefore\ \frac{dA}{dx}=2x.$$

$$\therefore\ \frac{dA}{dx}=9{\cdot}4 \text{ (when } x=4{\cdot}7).$$

But
$$\frac{\delta A}{\delta x}=\frac{dA}{dx} \text{ (very approx.).}$$

$$\therefore\ \frac{\delta A}{\delta x}=9{\cdot}4 \text{ (very approx.).}$$

$$\therefore\ \delta A=9{\cdot}4\delta x \text{ (very approx.).}$$

Also
$$\begin{aligned}\delta x&=4\% \text{ of } 4{\cdot}7\\&=0{\cdot}188.\end{aligned}$$

$$\begin{aligned}\therefore\ \delta A&=9{\cdot}4\times 0{\cdot}188 \text{ (very approx.)}\\&=1{\cdot}8 \text{ (very approx.).}\end{aligned}$$

Hence the area is almost 2 sq. cm. greater than it appears to be, as the result of the calculations based on the measurements.

Example 2. *The time of swing of a simple pendulum varies as the square root of the length, and at a certain place the pendulum that beats 1 second is 1 metre long. Find the time of swing of a pendulum 1·02 m. long.*

Let $t =$ time of swing in seconds,

and $l =$ length of pendulum in metres.

Since t varies as the square root of l, $\therefore \dfrac{t}{\sqrt{l}}$ is constant in value.

But t is 1 second when l is 1 metre.

$$\therefore \frac{t}{\sqrt{l}} = \frac{1}{\sqrt{1}} = 1.$$

$$\therefore t = \sqrt{l}.$$

$$\therefore \frac{dt}{dl} = \frac{1}{2\sqrt{l}}.$$

$$\therefore \frac{dt}{dl} = \frac{1}{2} \text{ (when } l=1\text{)}.$$

But $$\frac{\delta t}{\delta l} = \frac{dt}{dl} \text{ (very approx.)}.$$

$$\therefore \frac{\delta t}{\delta l} = \frac{1}{2} \text{ (very approx.) when } l=1 \quad\text{.....................(1)}$$

$$\therefore \delta t = \tfrac{1}{2}\delta l \text{ (very approx.)}.$$

Also $$\delta l = 0·02 \text{ metres}.$$

$$\therefore \delta t = 0·01 \text{ seconds}.$$

Hence the time of swing is 1·01 seconds (very approx.).

Example 3. *A line receives a small increase of p %. Find the corresponding percentage increase in the volume of the cube of which it is an edge.*

We have $$V = x^3.$$

$$\therefore \frac{dV}{dx} = 3x^2.$$

$$\therefore \frac{\delta V}{\delta x} = 3x^2 \text{ (very approx.)}.$$

$$\therefore \delta V = 3x^2\delta x \text{ (very approx.)} \quad\text{.........................(1)}$$

Now $$\delta x = \frac{px}{100} \quad\dots\dots(2)$$

$$\therefore\ \delta V = \frac{3px^3}{100} \text{ by (1) and (2).}$$

$$\therefore\ \frac{\delta V}{x^3} = \frac{3p}{100}.$$

$$\therefore\ \frac{\delta V}{V} = \frac{3p}{100}.$$

Hence the percentage increase in the volume is $3p$ %.

Example 4. *A man, in using the formula* $\frac{1}{2}ab \sin C$ *to find the area of a triangle, obtains a and b correctly as* 8·2 *cm. and* 4·0 *cm. respectively. His measurement of* C *is* 37° 15′, *which is* 2 *minutes too large. Find the resulting error in the calculation of the area.*

We have $$\Delta = \tfrac{1}{2}ab \sin C.$$

$$\therefore\ \frac{d\Delta}{dC} = \frac{\pi}{180} \times \frac{ab}{2} \cos C$$

$$= \frac{\pi}{180} \times \frac{8{\cdot}2 \times 4{\cdot}0}{2} \cos 37^\circ\, 15' \text{ (from the data of the problem)}$$

$$= 0{\cdot}228 \quad\dots\dots(1)$$

Now $$\frac{\delta\Delta}{\delta C} = \frac{d\Delta}{dC} \text{ (very approx.)}$$

$$= 0{\cdot}228 \text{ (very approx.) by (1).}$$

$$\therefore\ \delta\Delta = 0{\cdot}228\,\delta C \text{ (very approx.).}$$

Also $$\delta C = -2'$$

$$= -\frac{1}{30} \text{ degree.}$$

$$\therefore\ \delta\Delta = -0{\cdot}0076 \text{ sq. cm. (very approx.).}$$

Example 5. *In using a tangent-galvanometer, the current that passes through is proportional to the tangent of the angle of deflection. If the current is increased* 2%, *find the percentage increase of the angle of deflection, when the angle is* θ *radians.*

Let C denote the strength of the current, and θ denote the angle of deflection measured in radians.

$$\therefore\ \frac{C}{\tan\theta} = \text{constant.}$$

$$\therefore \frac{d}{d\theta}\left(\frac{C}{\tan\theta}\right)=0.$$

$$\therefore \tan\theta\frac{dC}{d\theta}-C\sec^2\theta=0.$$

$$\therefore \frac{dC}{d\theta}=\sec\theta\,\operatorname{cosec}\theta C.$$

$$\therefore \frac{dC}{d\theta}=\frac{2C}{\sin 2\theta}\text{.......................................}(1)$$

Now $$\frac{\delta C}{\delta\theta}=\frac{dC}{d\theta}\text{ (very approx.).}$$

$$\therefore \frac{\delta C}{\delta\theta}=\frac{2C}{\sin 2\theta}\text{ (very approx.) by (1).}$$

$$\therefore \delta\theta=\frac{\sin 2\theta}{2C}\delta C\text{ (very approx.)......................}(2)$$

But $$\delta C=0{\cdot}02C\text{ ....}(3)$$

$$\therefore \delta\theta=0{\cdot}01\sin 2\theta\text{ by (2) and (3).}$$

$$\therefore \frac{\delta\theta}{\theta}=0{\cdot}01\frac{\sin 2\theta}{\theta}.$$

∴ there is a percentage increase of $\frac{\sin 2\theta}{\theta}$ in θ, when θ is measured in radians.

5. More than one source of error.

THE PRINCIPLE OF THE ADDITION OF SMALL ERRORS.

Frequently there is more than one source of error in the formula we are manipulating. Thus, if we are measuring the sides a and b of a rectangle and calculating its area from the formula $A=a\times b$, there are *two* sources of error, viz. the error made in measuring a and the error made in measuring b.

Again, if we are measuring two sides a, b and the contained angle C of a triangle, and using the formula $\Delta=\frac{1}{2}ab\sin C$ to calculate its area, there are *three* sources of error, viz. the two errors made in measuring the sides and the error made in measuring the contained angle.

In all such cases, if the error be small, *we treat each source of error separately and add the results.*

A complete mathematical discussion of the above principle of "the addition of small errors" would be too hard for this work, but a few illustrations and diagrams will make the matter plain.

Rectangle.

Let OAPB be a rectangle whose sides have the lengths

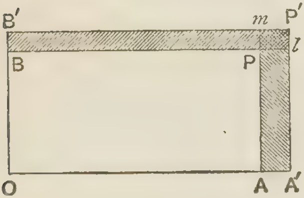


FIG. 148.

$OA = a$ cm.,

$OB = b$ cm.

Let OA'P'B' be the rectangle actually obtained, so that there is an error $AA' \equiv \delta a$ in OA and an error $BB' \equiv \delta b$ in OB.

Let S sq. cm. denote the area of the rectangle.

∴ $(S + \delta S)$ sq. cm. will denote the area of the rectangle OA'P'B'.

Hence δS is the error in the actual area of the rectangle, and is represented by the shaded part.

Now the shaded part δS involves three parts:

(1) The rectangle $AA'lP$.
(2) The rectangle $BPmB'$.
(3) The rectangle $PlP'm$.

Now plainly (3) is very small compared with either (1) or (2). Hence, if we neglect (3), we see that

$$\delta S = AA'lP + BPmB' \text{ (very approx.).}$$

But $AA'lP$ is the error due to the error in a, it being supposed that b is correct; and BP in B' is the error due to the error in b, a being supposed correct.

Hence, to find the error in the area of the rectangle, we estimate the error in the rectangle due to the error in a only, b being considered correct; we thereafter estimate the error in the rectangle due to the error in b only, a being considered correct. To find the error in the area S of the rectangle, we add these two errors together.

A few examples will make the method of procedure plain, when there is more than one source of error.

Example 1. *A man measures the sides of a rectangle with a ruler which can measure correctly only to two places of decimals. The results he obtains are* 7·93 *cm. and* 3·48 *cm. Find the largest error which he can make in the calculation of the area from the formula* $A = a \times b$.

Let $\delta_1 A$ and $\delta_2 A$ denote the errors in A due to the errors in a and b respectively.

Since the ruler can measure correct to two places of decimals, there is a possible maximum error of 0·005 (of excess or defect) in both a and b.

Hence $\delta a = 0{\cdot}005$

and $\delta b = 0{\cdot}005$.

Error in measuring a.

We differentiate $A = ab$, keeping b constant, since b is assumed to be correct in dealing with the error in a, *i.e.* $\frac{dA}{da} = b$.

Hence $\frac{\delta_1 A}{\delta a} = \frac{dA}{da} = b = 3{\cdot}48$ (very approx.).

$$\therefore\ \delta_1 A = 3{\cdot}48\delta a = 3{\cdot}48 \times 0{\cdot}005 = 0{\cdot}0174 \text{ (very approx.)}.$$

Error in measuring b.

As before, $\frac{dA}{db} = a$, since in differentiating $A = ab$, in dealing with the error in b, we regard a as correct and hence constant in differentiating with regard to b.

$$\therefore\ \frac{\delta_2 A}{\delta b} = \frac{dA}{db} = a = 7{\cdot}93 \text{ (very approx.)}.$$

$$\therefore\ \delta_2 A = 7{\cdot}93\delta b = 7{\cdot}93 \times 0{\cdot}005 = 0{\cdot}0397 \text{ (very approx.)}.$$

Hence the maximum error in $A \equiv \delta_1 A + \delta_2 A = 0{\cdot}0174 + 0{\cdot}0397$

$$= 0{\cdot}057 \text{ (approx.)}$$
$$= 0{\cdot}06 \text{ sq. cm.}$$

Example 2. *A man measures the sides a, b of a right-angled triangle with a view to calculating the hypotenuse c. If he can measure only correctly to two places of decimals and finds that the sides are* $a = 7{\cdot}93$ *cm. and* $b = 5{\cdot}42$ *cm., find the greatest possible error in his calculation of c.*

We have
$$\begin{aligned} c^2 &= a^2+b^2 \\ &= (7{\cdot}93)^2+(5{\cdot}42)^2 \\ &= 92{\cdot}2613. \\ \therefore\ c &= 9{\cdot}6053 \quad \ldots\ldots (1) \end{aligned}$$

Let $\delta_1 c$ and $\delta_2 c$ be the errors in c due to the errors in a and b respectively.

Error in a.

In making allowances for the error in a, we treat b as correct and hence constant in differentiating $c^2=a^2+b^2$.

Hence
$$\begin{aligned} \frac{dc}{da} &= \frac{a}{c} \\ &= \frac{7{\cdot}93}{9{\cdot}61}. \end{aligned}$$

$$\therefore\ \frac{\delta_1 c}{\delta a}=\frac{dc}{da}=\frac{7{\cdot}93}{9{\cdot}61} \text{ (very approx.).}$$

$$\begin{aligned} \therefore\ \delta_1 c &= \frac{7{\cdot}93}{9{\cdot}61}\times 0{\cdot}005\ (\because\ \delta a=0{\cdot}005) \\ &= 0{\cdot}0041 \text{ cm.} \quad \ldots\ldots (2) \end{aligned}$$

Error in b.

We have
$$\begin{aligned} \frac{dc}{db} &= \frac{b}{c} \text{ (treating } a \text{ as constant)} \\ &= \frac{5{\cdot}42}{9{\cdot}61}. \end{aligned}$$

$$\therefore\ \frac{\delta_2 c}{\delta b}=\frac{dc}{db}=\frac{5{\cdot}42}{9{\cdot}61} \text{ (very approx.).}$$

$$\begin{aligned} \therefore\ \delta_2 c &= \frac{5{\cdot}42}{9{\cdot}61}\times 0{\cdot}005\ (\because\ \delta b=0{\cdot}005) \\ &= 0{\cdot}0028 \text{ cm.} \quad \ldots\ldots (3) \end{aligned}$$

Hence the greatest possible error in $c \equiv \delta_1 c+\delta_2 c=0{\cdot}007$ cm.

Example 3. *A man finds the height of a tower by finding α° the angle of elevation of its summit and the distance c ft. of his position from the foot of the tower. If he finds α to be $37{\cdot}5^\circ$ and c to be 97·7 ft., and if he can make a 3% error in either observation, find the highest percentage error which he can make in his result.*

Let h be the height of the tower in feet.

$$\therefore\ h = c \tan \alpha,$$

assuming the measurements to be strictly accurate.

$$\begin{aligned} i.e.\ h &= 97{\cdot}7 \times \tan 37{\cdot}5^\circ \text{ ft.} \\ &= 97{\cdot}7 \times 0{\cdot}7673 \text{ ft.} \\ &= 74{\cdot}9652 \text{ ft.} \end{aligned}$$

Let $\delta_1 h$ and $\delta_2 h$ be the errors in h due to the errors in c and α respectively.

Error in measuring c.

We differentiate $h = c \tan \alpha$, keeping α constant, since α is assumed to be correct in dealing with the error in c.

Hence
$$\begin{aligned} \frac{dh}{dc} &= \tan \alpha \\ &= 0{\cdot}7673. \end{aligned}$$

$$\therefore\ \frac{\delta_1 h}{\delta c} = \frac{dh}{dc} = 0{\cdot}7673 \text{ (very approx.).}$$

$$\therefore\ \delta_1 h = 0{\cdot}7673\delta c \text{ (very approx.).}$$

$$\begin{aligned} \therefore\ \delta_1 h &= 0{\cdot}7673 \times 0{\cdot}03 \times 97{\cdot}7 \ (\because\ \delta c = 3\% \text{ of } 97{\cdot}7) \\ &= 2{\cdot}2490. \end{aligned}$$

Error in measuring α.

We differentiate $h = c \tan \alpha$, keeping c constant, because c is assumed to be correct in dealing with the error in α.

Hence
$$\begin{aligned} \frac{dh}{d\alpha} &= \frac{\pi}{180} c \sec^2 \alpha \\ &= 0{\cdot}01745 \times 97{\cdot}7 \times (1{\cdot}2605)^2 \\ &= 2{\cdot}7087. \end{aligned}$$

$$\therefore\ \frac{\delta_2 h}{\delta \alpha} = \frac{dh}{d\alpha} = 2{\cdot}7087 \text{ (very approx.).}$$

$$\therefore\ \delta_2 h = 2{\cdot}7087\delta\alpha \text{ (very approx.).}$$

$$\begin{aligned} \therefore\ \delta_2 h &= 2{\cdot}7087 \times 0{\cdot}03 \times 37{\cdot}5 \\ &= 3{\cdot}0473. \end{aligned}$$

$$\begin{aligned} \therefore\ \text{Highest possible error} \equiv \delta_1 h + \delta_2 h &= 2{\cdot}2490 + 3{\cdot}0473 \\ &= 5{\cdot}2963 \text{ ft.} \\ &= 5 \text{ ft. (very approx.)} \end{aligned}$$

$$\text{and the highest possible percentage error} = \frac{5{\cdot}2963 \times 100}{74{\cdot}9652}\,\%$$
$$= 7{\cdot}065\%$$
$$= 7\%\ \text{(very approx.)}.$$

Example 4. *The load W lb. which can be supported at the middle of an oaken beam b inches broad, d inches deep, and l inches long is given by the formula* $W = 840\,\dfrac{bd^2}{l}$. *Find the load that can be supported by a beam 10 ft. long, 6 inches broad, and 8 inches deep, and find the heaviest weight that it will be safe to hang on if the above measurements can be relied on as being correct only to within a possible error of 2%.*

Here $l = 120, \quad b = 6, \quad d = 8,$

and $\therefore\ W = 840 \times \dfrac{6 \times 8^2}{120} = 2688$ lb.

Let $\delta_1 W$, $\delta_2 W$, $\delta_3 W$ be errors due to the errors δl, δb, δd in length, breadth, and depth respectively.

Then $\dfrac{\delta_1 W}{\delta l} = \dfrac{dW}{dl} = 840\dfrac{bd^2}{l^2} = \dfrac{7 \times 8^2}{20} = 22{\cdot}4$ (very approx.).

$\therefore\ \delta_1 W = 0{\cdot}02 \times 120 \times 22{\cdot}4 = 53{\cdot}76$ (since δl represents a possible error of 2% of l).

Also $\dfrac{\delta_2 W}{\delta b} = \dfrac{dW}{db} = 840\dfrac{d^2}{l} = 7 \times 8^2 = 448$ (very approx.).

$\therefore\ \delta_2 W = 0{\cdot}02 \times 6 \times 448 = 53{\cdot}76$ (taking δb as 2% of b).

Similarly $\dfrac{\delta_3 W}{\delta d} = \dfrac{dW}{dd} = 840\dfrac{2bd}{l} = 14 \times 6 \times 8 = 672$ (very approx.).

$\therefore\ \delta_3 W = 0{\cdot}02 \times 8 \times 672 = 107{\cdot}52$ (taking δd as 2% of d).

$\therefore\ \delta W = \delta_1 W + \delta_2 W + \delta_3 W = 53{\cdot}76 + 53{\cdot}76 + 107{\cdot}52 = 215{\cdot}04.$

$\therefore$ Heaviest weight it will be safe to hang $= 2688 - 215$
$= 2473$ lb. (approx.).

6. The Method of Infinitesimals.

Finding differential coefficients directly, by means of pure geometry.

We shall now use the methods of this chapter to show how to find directly by pure geometry the differential coefficients of the trigonometrical functions. We shall frequently require the following principle :—

Let OP and OQ be two lines meeting at O. Let OP be very nearly equal to OQ, and let the angle POQ be very small. With centre O and radius OP, describe the very small arc of a circle PN. Then plainly we may treat the *very short arc* PN as a straight line perpendicular to OQ; since a circle cuts every one of its radii at right angles.

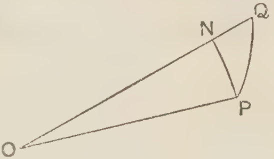


FIG. 149.

Example 1. *To find by pure geometry the differential coefficient of tan θ where θ is measured in radians.*

Let OH be a line of unit length, and let HK be drawn perpendicular to OH. Let OP be drawn making $\hat{HOP} = \theta$ radians. Let OQ be drawn making $\hat{HOQ} = (\theta + \delta\theta)$ radians, where $\delta\theta$ is very small. With centre O and radius OP draw the arc PN. Let $HP = y$, and $HQ = y + \delta y$.

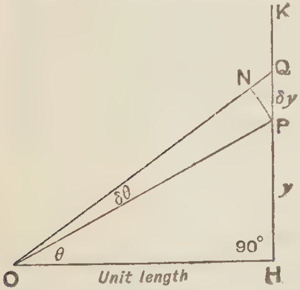


FIG. 150.

Plainly $y = \tan\theta$

and $y + \delta y = \tan(\theta + \delta\theta)$.

We wish to find geometrically $\frac{\delta y}{\delta\theta}$.

Consider the small right-angled triangle PNQ.

We have $PQ = \delta y$ ....(1)

and $PN = OP \times \delta\theta$ ($\because$ arc PN subtends the angle $\delta\theta$ at the centre of the circle),

i.e. $PN = \sec\theta\,\delta\theta \left(\because \frac{OP}{OH} = \sec\theta \text{ and } OH = 1\right)$ ....(2)

Also $\frac{PN}{PQ} = \sin \hat{NQP} = \cos(\theta + \delta\theta)$..............................(3)

$$\therefore \frac{\sec\theta\,\delta\theta}{\delta y} = \cos(\theta + \delta\theta), \qquad i.e. \ \frac{\delta y}{\delta\theta} = \sec\theta \sec(\theta + \delta\theta).$$

Hence, proceeding to the limit in the usual way, we get

$$\frac{dy}{d\theta} = \sec^2\theta,$$

$$i.e. \ \frac{d \tan\theta}{d\theta} = \sec^2\theta \ (\theta \text{ in radians}).$$

Example 2. *To find by pure geometry the differential coefficient of* $\sin\theta$ *where* θ *is measured in radians.*

Let the figure represent a circle of unit radius. Let OP, OQ make angles θ, $\theta+\delta\theta$ respectively with OA where $\delta\theta$ is very small. Draw PM, QN each perpendicular to OA, and PH perpendicular to QN. Let $PM=y$, and $QN=y+\delta y$.

We then have $y=\sin\theta$

$$y+\delta y=\sin(\theta+\delta\theta).$$

We wish to find geometrically $\frac{\delta y}{\delta\theta}$.

Consider the very small right-angled triangle PHQ, in which the arc PQ of the circle may be treated as a straight line, since it is very short.

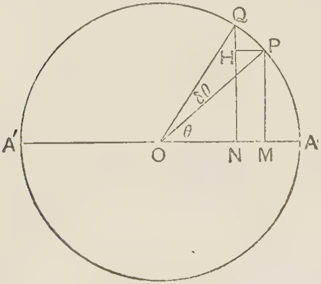


FIG. 151.

We have $QH=\delta y$

$$PQ=\delta\theta.$$

Also $$\frac{QH}{PQ}=\sin Q\hat{P}H$$

$=\cos H\hat{P}O$ (very approx.) (since $Q\hat{P}O$ is a right angle very approximately, because the arc of a circle cuts a radius perpendicularly).

$$\therefore \frac{QH}{PQ}=\cos\theta \quad \because \text{HP is parallel to OA,}$$

$$\textit{i.e. } \frac{\delta y}{\delta\theta}=\cos\theta \text{ (very approx.),}$$

$$\textit{i.e. } \frac{dy}{d\theta}=\cos\theta \text{ in the limit.}$$

$$\therefore \frac{d\sin\theta}{d\theta}=\cos\theta \text{ as before.}$$

[*N.B.*—The above method is usually called the "*method of infinitesimals*," because the old name for "*small quantities*" like δx, δy, $\delta\theta$ was "*infinitesimals.*"]

EXAMPLES XIV.

1. The following are details regarding rates raised in England and Wales per head of population :—

1902–3	1903–4	1904–5	1905–6	1906–7	1907–8	1908–9
£1 10 7	£1 11 10	£1 13 4	£1 14 3	£1 14 8	£1 14 4	£1 14 11

If t denote the time and R the amount of the rates, calculate $\frac{\delta R}{\delta t}$ for each interval, and hence tabulate the rate at which rates were increasing in 1903, 1904, 1905, 1906, 1907, 1908.

2. Deduce as approximately as you can the value of $\frac{dy}{dx}$ from the following table of values, by estimating the mean value of $\frac{\delta y}{\delta x}$ in the two intervals and taking the mean of these two results :—

x	4·73	5·80	5·87
y	27·10	39·44	40·33

3. Proceed as in the previous question and find the value approximately of $\frac{dl}{dt}$ from the following data :—

t	5·79	5·82	5·85
l	2·406	2·412	2·419

4. Deduce as approximately as you can the value of $\frac{dy}{dx}$ when $x=2·39$ from the following table :—

x	2·34	2·39	2·44
y	0·73	0·82	0·85

5. The motion of a particle sliding down an inclined plane is observed. It is noted that 3, 4, 5 seconds respectively from the beginning of motion the distances traversed are 7·2, 12·8, 20·0 feet respectively. Find as approximately as you can the speed of the particle after 4 seconds.

6. In the following table C denotes the radio-activity of a substance, t hours after the observations were commenced. There is reason for believing that $\frac{dC}{dt}=aC$, where a is a constant.

Try if this is so, and, if so, find the most probable value of a.

t	0	7·9	11·8	23·4	29·2	32·6	49·2	62·1	71·4
C	100	64	47·4	19·6	13·8	10·3	3·7	1·86	0·86

(Bd. of Educ.)

7. Assuming that you wish to obtain experimentally the formula connecting $\frac{dy}{dx}$ with x where $y=\sqrt{x}$, look up your square root tables, and calculate the mean value of $\frac{\delta y}{\delta x}$ for the x-intervals as given in the following table :—

x	5·50	5·51	5·93	5·94	6·82	6·83	7·53	7·54	8·66	8·67	8·90	8·91	9·46	9·47	9·89	9·90	10·32	10·
y	2·345	2·347																
δy	0·002																	
δx	0·01																	
$\frac{\delta y}{\delta x}$	0·2																	

Hence calculate $\sqrt{x}\times\frac{dy}{dx}$ and state what result you deduce.

8. If $y=\cos x°$, calculate as accurately as you can $\frac{dy}{dx}$ for the follow-

ing values of x, directly from your table of cosines, by filling in the subjoined table :—

x	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	120°	140°	160°
$\frac{\delta y}{\delta x}$													

(See Art. 2, Ex. 1.)

Hence evaluate $\operatorname{cosec} x^\circ \times \frac{dy}{dx}$ for each of the above values of x by consulting the cosecant tables. Deduce the formula for $\frac{dy}{dx}$ in terms of x which you obtain.

9. If $y=10^x$, calculate and fill in from your anti-logarithm tables the subjoined table :—

x	0·10	0·20	0·30	0·40	0·50	0·60	0·70	0·80	0·90	1·00
y										
$\frac{\delta y}{\delta x}$										

(See Art. 2, Ex. 2.)

Hence evaluate $\frac{dy}{dx} \div y$, and so deduce the law for $\frac{dy}{dx}$, giving the required formula.

10. If $y=\log_{10} \sin x^\circ$, calculate and fill in from your table of logarithmic sines the subjoined table :—

x	10°	20°	30°	40°	50°	60°	70°	80°	90°
$\frac{\delta y}{\delta x}$									

Hence, by consulting your table of natural tangents, see if you can find the law for evaluating $\frac{d}{dx} \log_{10} \sin x^\circ$.

11. If $y = \tan x$ (x measured in degrees), calculate $\frac{\delta y}{\delta x}$ when $x = 27°$. Deduce the values of δy when $\delta x = 0·3°, 0·02°, 4', 8''$.

12. If $h = \cot \theta$ (θ measured in radians), calculate $\frac{\delta h}{\delta \theta}$ when $\theta = 0·62$ radians. Deduce the value of δh when $\delta\theta = 0·02, 0·03, 0·07, 0·09$ radians respectively.

13. A certain set of trigonometrical tables give the functional values only for a whole number of degrees. Use the principle of infinitesimals to calculate sin 47·2°.

14. A man makes an error of $p\%$ in measuring an angle as $\theta°$. Find the percentage error which he makes in deducing therefrom the values of its sine and tangent.

15. An error of 2 mm. is made in measuring x as 7·3 cm. Find the consequent error in the evaluation of $x + x^2$.

16. A man in measuring a line makes an error of 2%. If he finds the line to be 7·35 cm., find the percentage error in the area of the square described thereon.

[Use the formula $A = x^2$.]

17. In the previous question, if the man uses the same measurement to calculate the volume of the cube described with the given line as edge, find the percentage error he makes in his computation of the cube.

[Use the formula $V = x^3$.]

18. Considering the earth as a perfect sphere, a rope is 2 yards longer than would be necessary to pass round the equator in contact with the ground. How high must it be lifted uniformly from the ground so that it may be taut when its two ends are in contact?

19. A telegraph cable 100 miles long lies on a level stretch of desert. How much longer must an overhead line be, carried on poles 20 feet from the ground?

[*Hint.*—Arc of a circle = radius × circular measure of an angle.
Earth's radius = 4000 miles, approximately.]

20. If when iron is heated from 0° C. to 10° C., 1 linear foot expands to $(1 + \alpha)$ feet, where α is small, show that 1 cub. ft. of iron when heated from 0° C. to 10° C. will expand to $(1 + 3\alpha)$ cub. ft. Hence, if $\alpha = 0·00012$, find how many times heavier 1 cub. ft. of iron at 0° C. is than 1 cub. ft. of iron at 10° C.

21. An error of 3% is made in measuring an angle θ as $47{\cdot}7^\circ$. Find the percentage error in estimating the numerical value of $\sin\theta^\circ + \cos\theta^\circ$.

22. A man makes a small error of $p\%$ in finding by measurement that a certain angle is $73{\cdot}8^\circ$. Find the percentage error in obtaining the value of its tangent from the tables.

23. The following is a table giving values of y in terms of x, and another giving values of u in terms of y. What is u when $x = 8{\cdot}3$?

x	y	y	u
7	14·914	15	0·8169
8	16·128	16	0·7118
9	17·076	17	0·5543

(Bd. of Educ.)

24. If $pv = 100t$, and $p = 3000$ when $t = 300$, find v.
If $p = 3010$, and $t = 302$, find the new v.
If the second set of values be called $3000 + \delta p$, $300 + \delta t$, $v + \delta v$, what is δv?

Now use the formula $\delta v = \left(\dfrac{dv}{dp}\right)\delta p + \left(\dfrac{dv}{dt}\right)\delta t$, and calculate δv in the new way. (Bd. of Educ.)

25. Fill up the following table from your logarithmic tables :—

x	10·0	10·1	10·2	10·3	10·4	10·5	10·6	10·7	10·8	10·9	11·0
$\log_{10} x$											

Find as accurately as you can the values of $\dfrac{d\log_{10} x}{dx}$ in each interval.

Look up the table of reciprocals and tabulate for $\dfrac{1}{x}$.

Plot $\dfrac{d\log_{10} x}{dx}$ and $\dfrac{1}{x}$ against each other on squared paper and see whether $\dfrac{d\log_{10} x}{dx} = \dfrac{c}{x}$. If so, give the value of c.

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26. Repeat the procedure of the previous question for the two sets of values—

(i) $x=24{\cdot}0, 24{\cdot}1, 24{\cdot}2, 24{\cdot}3, 24{\cdot}4, 24{\cdot}5, 24{\cdot}6, 24{\cdot}7, 24{\cdot}8, 24{\cdot}9, 25{\cdot}0$.
(ii) $x=37{\cdot}0, 37{\cdot}1, 37{\cdot}2, 37{\cdot}3, 37{\cdot}4, 37{\cdot}5, 37{\cdot}6, 37{\cdot}7, 37{\cdot}8, 37{\cdot}9, 38{\cdot}0$.

27. Determine the value of c obtained in the previous questions, and then determine a more accurate value by using "Seven Figure Logarithm Tables."

28. Ox and Oy are two perpendicular lines. A is a point on Ox such that OA is 30 cm. P is a point on Oy whose distance OP has to be measured. It is desired to calculate AP thereafter. If OP is measured and found to be 57·6 cm. and there is an error of 0·3 cm. in the measurement, find the error in computing AP by Pythagoras' Theorem.

29. A certain metal expands by 0·0000103 of its length for every degree centigrade that the temperature rises. Now it is known that the time of beat of a pendulum varies as the square root of its length. Calculate the fraction of itself that the time of beat of a pendulum of the above metal increases if the temperature rises 2° C.

[*Hint.*—Show that $\frac{\delta t}{t}=\frac{\delta l}{2l}$, where t = time of beat in seconds and l = length of pendulum in centimetres.]

30. The area of a triangle ABC is to be calculated from the formula $\Delta=\frac{1}{2}ab\sin C$. The wooden protractor used for measuring purposes measures lines correctly, but diminishes angles by 2%. Find the consequent percentage error in the areas thus obtained.

If $a=7$ cm., $b=8$ cm., $C=37{\cdot}5°$, calculate the percentage error in the area.

31. A ruler which exaggerates all lengths by 3% is used to measure a rectangle whose sides are 17 cm. and 11 cm. Find the percentage error in the area calculated therefrom.

32. A man stands at a distance of 67 yds. from a tower and notes that its elevation is 52°. If he can measure distances correct to the nearest foot, and angles correct to the nearest degree, what is the greatest percentage error that he can make in estimating the height of the tower from the formula Height = Distance × (Tangent of elevation)?

33. A motor car running between two towns 59·25 miles apart is believed to be running at an average speed of 23 miles per hour. What is the error in the calculated time of running if there is an error of 0·7% in the average speed?

34. The diameter of a ball is found by using callipers to be 32·7 mm. What is the error made in calculating its circumference by taking $\pi = 3{\cdot}142$ instead of 3·14159?

35. The length of a ladder is calculated from the known fact that it reaches a window sill usually considered to be 30 ft. from the ground when its foot is 3 ft. from the wall of the house. What is the error in the length of the ladder if the true height of the window sill is 30 ft. 5 in.?

36. A boy flying a kite has paid out 150 yds. of string, and it seems to him that the string makes an angle of 45° with the horizon. Thence he calculates the height of the kite. What is the error in this calculation if the true angle made with the horizon is 45° 30′?

37. A boy calculates the height of a cliff by dropping a stone from the top and timing its fall by his watch, and then using the formula $s = 16t^2$. What error does he make if he takes 4 sec. instead of the true time $4\frac{1}{5}$ sec.?

38. A boy whose eye is 5 ft. from the ground stands at a distance of 12 ft. from the foot of a tree, and with a clinometer measures the angle of elevation of the top of the tree. He finds it to be 65°. If he makes an error of 0·5% in defect in measuring the angle, what percentage error does he make in calculating the height of the tree?

39. A tree whose height has been measured and found to be 25 ft. throws a shadow of 32 ft. in length. Thence is calculated the angle of elevation of the sun. If the true height of the tree is 25 ft. 2 in., what error is made in calculating the angle?

40. In a tangent galvanometer, if n is the number of turns of wire, r the mean radius of the coil, θ the deflection of the needle, and $H = 0{\cdot}18$, then if a current of ι amperes flows

$$\iota = \frac{10Hr}{2\pi n}\tan\theta.$$

In a galvanometer of 50 turns the mean radius is 20 cm. and the deflection is read off as 35°. What is the percentage error in the calculated current if 35·25° is the true deflection?

41. If in the same galvanometer the deflection is correctly read as 57°, what is the percentage error in the calculated current if the mean radius of the coil is 19·8 cm.?

42. A wire whose resistance is R ohms is made into a coil and placed in a calorimeter containing M grams of water. A current of C amperes flows through it for t seconds, and the rise of temperature is θ degrees centigrade.

We know that $M\theta = \frac{C^2Rt}{4 \cdot 2}$.

In a certain case $C = 1 \cdot 5$, $R = 30$, $t = 510$, $M = 127$, and θ is calculated. What is the error of θ if C is accurately 1·49?

43. In another case $M = 150$, $\theta = 5$, $R = 22$, $t = 900$, and C is calculated. What is the error of C if θ is accurately 5·2?

44. In another case $M = 180$, $\theta = 7$, $C = 1 \cdot 7$, $t = 600$, and R is calculated. What is the error of R if C is accurately 1·72 and θ is accurately 7·2?

45. A cyclometer is constructed for a 28″ bicycle wheel, and π is taken as 3·1416. If 28·1″ is the true height of the wheel, what is the error of the cyclometer when it shows a mile?

46. In the last example, if the wheel is correctly 28″ but the cyclometer is constructed on the principle that $\pi = 3 \cdot 14$, what is the error of the cyclometer when it shows a mile?

47. A cylinder of iron is measured, and its length is found to be 7 cm. and the diameter of its circular section 6 cm. Its mass is 1550 grams. From these data the density of the iron is calculated. What is the error in the calculated density if a faulty ruler is used in measuring the lengths, the true length of 1 cm. on it being 1·003 cm.?

48. The two sides of a triangle are measured and found to be 7·3 in. and 5·4 in., and the included angle 57°. What is the maximum error in the calculated area if the tenths of inches are estimated?

49. A triangle has two sides measured by a brass ruler which is correct at 15° C. At the time of the measurement, however, the temperature was 20° C. The lengths shown by the ruler were 5·7 and 6·9 in. and the included angle was 72°. Taking the coefficient of expansion of brass as 0·0000175, what is the percentage error in the calculated area of the triangle?

50. A triangle has its two sides correctly measured as 77 and 95 mm., but there is an error in defect of 27 minutes in the size of the included angle, which is taken to be 54°. What is the percentage error in the calculated area of the triangle? [Use $\Delta = \frac{1}{2}ab \sin C$.]

51. If the sides of a triangle are a, b, c yds., and the angles opposite to them A, B, C degrees, then we know that $a^2 = b^2 + c^2 - 2bc \cos A$.

If b and c are measured as 73 and 84, and are both 1% too short, while A is correctly 58°, what is the percentage error in the calculated value of a?

52. If the above formula be used to calculate the angle A when $a=7000$ yds., $b=5000$ yds., $c=3000$ yds., what is the error in A when each length is 20 yds. too short?

53. What is the error when each length is 20 yds. too long?

54. What is the error when a is 20 yds. too short and b and c are both 20 yds. too long?

CHAPTER XV.

MISCELLANEOUS METHODS OF INTEGRATION.

1. In the present chapter will be explained some methods of dealing with expressions which cannot be directly integrated. Only practice will give the student the requisite facility.

2. **Useful principle in integration.**

Since $\dfrac{dx^3}{dx} = 3x^2$, it will often be found convenient in integration to write $dx^3 = 3x^2\,dx$. We give some further examples :—

Since $\dfrac{d(1+x^2)}{dx} = 2x$, $\quad \therefore d(1+x^2) = 2x\,dx$.

Since $\dfrac{d \sin 3x}{dx} = 3 \cos 3x$ (x in radians),

$$\therefore\ d(\sin 3x) = 3 \cos 3x\,dx.$$

Since $\dfrac{d \sin^2\theta}{d\theta} = 2 \sin\theta \cos\theta$ (θ in radians),

$$\therefore\ d(\sin^2\theta) = 2 \sin\theta \cos\theta\,d\theta.$$

We shall now give some immediate illustrations of the practical application of the above principle.

Example 1. *Evaluate* $\int (1+x^2)^{10}\,x\,dx$.

Since $\dfrac{d(1+x^2)}{dx} = 2x$, $\quad \therefore$ we may write $2x\,dx = d(1+x^2)$

i.e. $x\,dx = \frac{1}{2}d(1+x^2)$.

Hence $\displaystyle\int (1+x^2)^{10}\,x\,dx \equiv \tfrac{1}{2}\int (1+x^2)^{10}\,d(1+x^2) \left\{\text{of the form} \int y^{10}dy\right\}$

$$\equiv \frac{(1+x^2)^{11}}{22}.$$

Example 2. *Evaluate* $\int \sin 3\theta \, d\theta$ *(θ in radians).*

Since $\dfrac{d(3\theta)}{d\theta}=3$, $\therefore$ we may write $d(3\theta)=3\,d\theta$

$$\text{i.e. } d\theta=\tfrac{1}{3}d(3\theta).$$

Hence
$$\int \sin 3\theta \, d\theta \equiv \tfrac{1}{3}\int \sin 3\theta \, d(3\theta) \left\{\text{of the form} \int \sin\phi \, d\phi\right\}$$
$$\equiv -\tfrac{1}{3}\cos 3\theta.$$

Example 3. *Evaluate* $\int \cos^2 x \, dx$ *(x in radians).*

We have
$$\cos^2 x=\tfrac{1}{2}(1+\cos 2x).$$
$$\therefore \int \cos^2 x \, dx=\tfrac{1}{2}\int (1+\cos 2x)dx$$
$$=\tfrac{1}{2}\left(x+\frac{\sin 2x}{2}\right).$$

Example 4. *Evaluate* $\int \sin^3 x \, dx$ *(x in radians).*

By the well-known formula,
$$\sin 3x=3\sin x-4\sin^3 x.$$
$$\therefore \sin^3 x=\tfrac{1}{4}(3\sin x-\sin 3x).$$
$$\therefore \int \sin^3 x \, dx=\tfrac{1}{4}\int (3\sin x-\sin 3x)dx.$$
$$=\tfrac{1}{4}\left(-3\cos x+\frac{\cos 3x}{3}\right).$$

Example 5. *Evaluate* $\int \sin^2 x \cos x \, dx$ *(x in radians).*

We have $\dfrac{d\sin x}{dx}\equiv\cos x$, $\therefore \cos x \, dx\equiv d(\sin x)$.
$$\therefore \int \sin^2 x \cos x \, dx=\int \sin^2 x d(\sin x)$$
$$=\frac{\sin^3 x}{3}.$$

3. Change of Variable.

In many cases, the integration can be more easily performed if we change the variable. We give some examples.

Example 1. *Evaluate* $\int (1+x)^{\frac{3}{2}}\,dx$.

Let $1+x=y. \quad \therefore \frac{dy}{dx}=1.$

Hence for our purpose we may more conveniently write

$$dx=dy.$$

$$\therefore \int (1+x)^{\frac{3}{2}}\,dx = \int y^{\frac{3}{2}}\,dy$$

$$= \tfrac{2}{5}\,y^{\frac{5}{2}}$$

$$= \tfrac{2}{5}(1+x)^{\frac{5}{2}}.$$

Example 2. *Evaluate* $\int (1+x^2)^{2\cdot 3}\,x\,dx$.

Let $y=1+x^2. \quad \therefore \frac{dy}{dx}=2x.$

Hence for our purpose we may more conveniently write

$$2x\,dx=dy.$$

$$\therefore \int (1+x^2)^{2\cdot 3}\,x\,dx = \tfrac{1}{2}\int y^{2\cdot 3}\,dy$$

$$= \frac{y^{3\cdot 3}}{6\cdot 6}$$

$$= \frac{5}{33}(1+x^2)^{3\cdot 3}.$$

N.B.—*The above examples will make clear the justification by algebra of the principle used in Art. 2.*

Example 3. *Evaluate* $\int \sqrt{1-x^2}\,dx$.

Let $x=\sin\theta$ (θ in radians).

$$\therefore \frac{dx}{d\theta}=\cos\theta.$$

$$\therefore dx=\cos\theta\,d\theta.$$

$$\therefore \int \sqrt{1-x^2}\,dx = \int \cos\theta \,.\, \cos\theta\,d\theta$$

$$= \int \cos^2\theta\,d\theta$$

$= \tfrac{1}{2}(\theta+\sin\theta\cos\theta)$ (by Ex. 3, Art. 2, Chap. XV.)

$$= \tfrac{1}{2}(\sin^{-1}x + x\sqrt{1-x^2}).$$

Example 4. *Evaluate* $\int(1+x)\sqrt{1-x^2}\,dx$.

Let $x=\sin\theta$.

$$\therefore\ dx=\cos\theta\,d\theta.$$

$$\begin{aligned}\therefore\ \int(1+x)\sqrt{1-x^2}\,dx&=\int(1+\sin\theta)\,.\,\cos\theta\,.\,\cos\theta\,d\theta\\&=\int(\cos^2\theta+\cos^2\theta\sin\theta)d\theta\\&=\int\cos^2\theta\,d\theta+\int\cos^2\theta\sin\theta\,d\theta\\&=\int\cos^2\theta\,d\theta-\int\cos^2\theta\,d(\cos\theta)\\&=\tfrac{1}{2}(\theta+\sin\theta\cos\theta)-\frac{\cos^3\theta}{3}\\&=\tfrac{1}{2}\sin^{-1}x+\tfrac{1}{2}x\sqrt{1-x^2}-\tfrac{1}{3}(1-x^2)^{\frac{3}{2}}.\end{aligned}$$

4. Change of Variable in Definite Integrals.

We shall now show how to change the variable in the case of Definite Integrals, but first of all we must study the methods of finding the range in the case of one variable given the range in the case of the other variable.

Example 1. *Subjoined is the graph connecting y with x. Find the range covered by y as x varies from* 1 *to* 3.

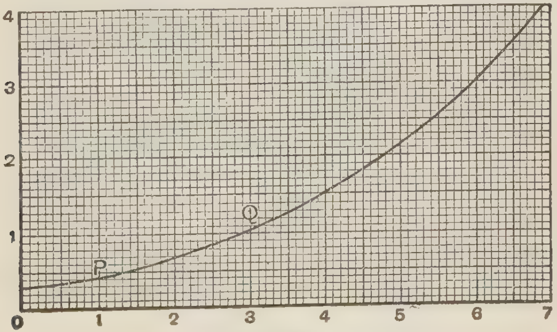

FIG. 152.

The points corresponding to $x=1$ and $x=3$ on the graph are P and Q. Thus, as x proceeds from 1 to 3 on the axis of x, the corresponding point on the curve travels from P to Q. Hence y travels from 0·41 to 1·05.

Similarly, as x travels from 0 to 7, y travels from 0·29 to 4·12.

Example 2. *Trace the graph of* $y=x+1$, *and hence read off from your graph the range traversed by* y *as* x *travels from* 1 *to* 4.

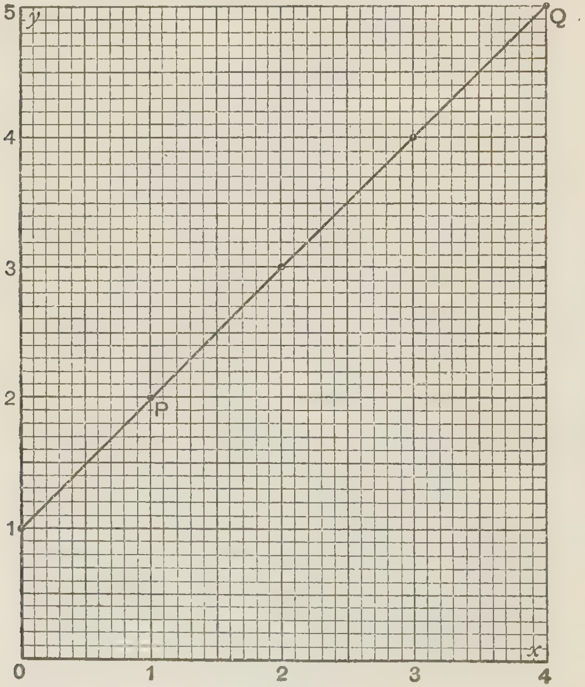

FIG. 153.

As x travels from 1 to 4 the corresponding point on the graph travels from P to Q, and consequently y travels from 2 to 5.

Example 3. *Trace the graph of $y = x^2 + 1$, and hence find the range traversed by y as x travels from 2·1 to 3·7.*

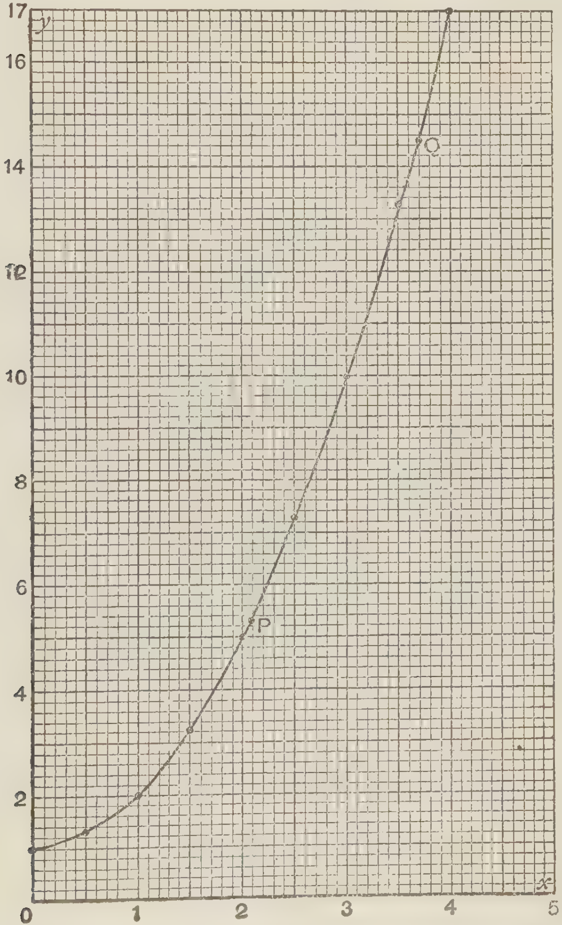

FIG. 154.

As x travels from 2·1 to 3·7 the corresponding point on the graph travels from P to Q, and consequently y travels from 5·4 to 14·5.

Example 4. *Trace the graph of* $y=\tan^{-1} x$ *(measured in radians), and hence obtain the range of* y *as* x *travels from* (1) 0 *to* 1 ; (2) *from* 0 *to* ∞.

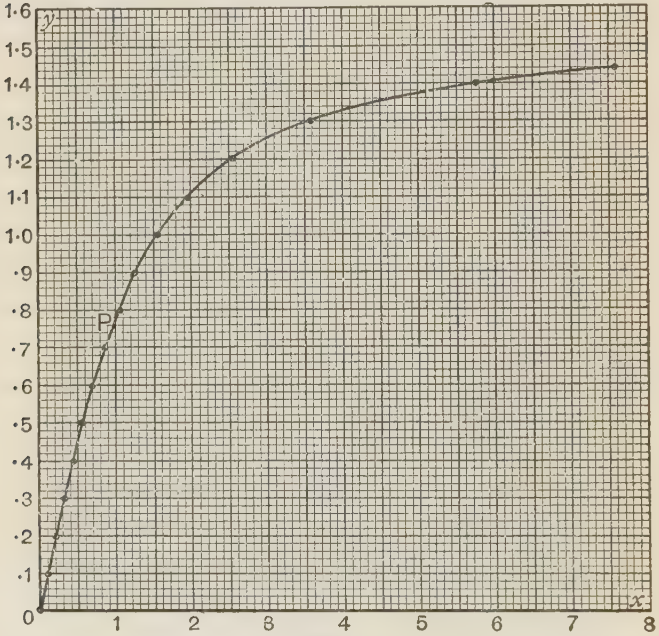

FIG. 155.

(1) As x travels from 0 to 1 the corresponding point on the graph travels from 0 to P, and consequently y travels from 0 to 0·79 $\left(i.e.\ \frac{\pi}{4}\right)$.

(2) As x travels from 0 to ∞, the corresponding point on the graph travels from 0 to ∞, and consequently y travels from 0 to $\frac{\pi}{2}$, as is clear from the tabulated value of $\tan y$ as y approches $\frac{\pi}{2}$

(because, the larger x becomes, the more closely does the curve approach the line $y=\frac{\pi}{2}$).

5. We shall now apply these principles to the evaluation of some Definite Integrals.

Example 1. *Evaluate* $\int_0^{0\cdot 8}\sqrt{1-x^2}\,dx$.

Let $x=\sin\theta$ (θ in radians).

Then, as x varies from 0 to 0·8,

$\therefore$ $\sin\theta$ varies from 0 to 0·8,

that is, θ varies from 0 to 0·927 radians.

$$\therefore \int_0^{0\cdot 8}\sqrt{1-x^2}\,dx=\int_0^{0\cdot 927}\cos^2\theta\,d\theta$$

$$=\frac{1}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{0\cdot 927}$$

$$=\frac{1}{2}\left[0\cdot 927+\frac{0\cdot 960}{2}\right]$$

$=0\cdot 70$ (correct to two places of decimals).

Example 2. *Evaluate* $\int_0^1 x\sqrt{1-x^2}\,dx$.

Let $x=\sin\theta$ (θ in radians).

$\therefore$ $dx=\cos\theta\,d\theta$..(1)

As x ranges from 0 to 1, θ ranges from 0 to $\frac{\pi}{2}$.........................(2)

$$\therefore \int x\sqrt{1-x^2}\,dx=\int_0^{\frac{\pi}{2}}\sin\theta\cos^2\theta\,d\theta$$

$$=-\int_0^{\frac{\pi}{2}}\cos^2\theta\,d(\cos\theta)\ (\because\ d\cos\theta=-\sin\theta\,d\theta)$$

$$=-\left[\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{3}.$$

Aliter.

$$\int_0^1 x\sqrt{1-x^2}\,dx = \frac{1}{2}\int_0^1 \sqrt{1-x^2}\,d(x^2)$$

$$= -\frac{1}{2}\int_0^1 (1-x^2)^{\frac{1}{2}}d(1-x^2)$$

$$= -\frac{1}{2}\left[\frac{(1-x^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$$

$$= \frac{1}{3}.$$

Example 3. *Evaluate* $\int_1^2 \frac{\sqrt{1+x^2}}{x^4}dx.$

Let $x = \tan\theta$ (θ in radians).

$$\therefore\ dx = \sec^2\theta\,d\theta \text{ ....(1)}$$

As x ranges from 1 to 2, θ ranges from $\frac{\pi}{4}$ to 1·107 radians.........(2)

$$\therefore \int_1^2 \frac{\sqrt{1+x^2}}{x^4}dx = \int_{\frac{\pi}{4}}^{1\cdot107} \frac{\sec\theta}{\tan^4\theta}\,.\,\sec^2\theta\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{1\cdot107} \frac{\cos\theta\,d\theta}{\sin^4\theta} = \int_{\frac{\pi}{4}}^{1\cdot107} (\sin\theta)^{-4}\,d\sin\theta$$

$$= \left[\frac{(\sin\theta)^{-3}}{-3}\right]_{\frac{\pi}{4}}^{1\cdot107}$$

$$= \tfrac{1}{3}[2\cdot828 - 1\cdot398]$$

$$= \tfrac{1}{3}[1\cdot430] = 0\cdot477.$$

Example 4. *Find the area enclosed between two parallel chords of a circle of radius 3 cm. at distances 1 and 2 cm. from the centre respectively.*

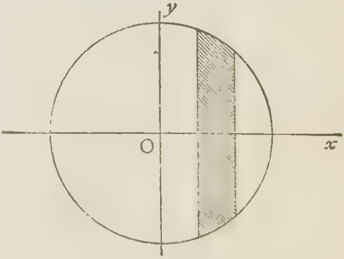

FIG. 156.

The area required

$$= 2\int_1^2 y\,dx$$

$$= 2\int_1^2 \sqrt{9-x^2}\,dx \ (\because\ x^2+y^2=9).$$

Let
$$x=3\cos\theta.$$
$$\therefore\ dx=-3\sin\theta\, d\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

As x ranges from 1 to 2, θ ranges from 1·231 to 0·841 radians....(2)

Hence area required

$$=2\int_{\cdot 231}^{0\cdot 841} 3\sin\theta\times(-3\sin\theta)d\theta$$

$$=-9\int_{1\cdot 231}^{0\cdot 841}(1-\cos 2\theta)d\theta$$

$$=-9\left[\theta-\frac{\sin 2\theta}{2}\right]_{1\cdot 231}^{0\cdot 841}$$

$$=9(1{\cdot}231-0{\cdot}841)+\frac{9}{2}(0{\cdot}9938-0{\cdot}6280)=5{\cdot}156 \text{ sq. cm.}$$

6. Integration by parts.

The following artifice is of great practical utility. It frequently happens that we are called upon to integrate $\int v\,du$ where u and v are functions of x, and that we cannot immediately evaluate $\int v\,du$, but we can write down at once the value of $\int u\,dv$. We therefore proceed thus :—

Since $d(uv)=u\,dv+v\,du$ (by Chap. XII., Art. 2),
$$\therefore\ v\,du=d(uv)-u\,dv.$$

Hence $\int v\,du=uv-\int u\,dv$

and the integration of $\int v\,du$ is immediately effected if we know $\int u\,dv$.

The resulting process is called the method of "*integrating by parts.*"

We shall now give some examples to illustrate its utility.

Example 1. *Integrate* $\int x \cos x\,dx$.

We have
$$\int x\cos x\,dx = \int x\,d(\sin x)$$
$$= x\sin x - \int \sin x\,dx$$
$$= x\sin x + \cos x.$$

Example 2. *Integrate* $\int x(1+x)^{100}dx$.

$$\int x(1+x)^{100}dx = \int x\,d\left\{\frac{(1+x)^{101}}{101}\right\}$$
$$= \frac{x(1+x)^{101}}{101} - \int \frac{(1+x)^{101}}{101}dx$$
$$= \frac{x(1+x)^{101}}{101} - \frac{(1+x)^{102}}{101\times 102}.$$

7. Note on negative areas.

In solving a definite integral, *e.g.* $\int_a^b f(x)\,dx$, it is advisable to sketch the graph $y=f(x)$, so as to know beforehand, by way of a rough check, what form of result to expect. In that way needless and obscure errors can often be avoided. For example, $\int_0^\pi \cos x\,dx = 0$. To explain this we trace the graph of $y=\cos x$, obtaining a graph of the form—

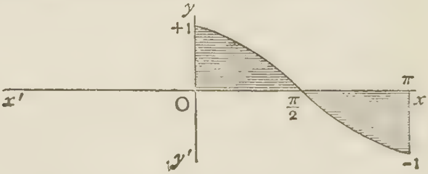

FIG. 157.

The area of the shaded part above the axis of x must be regarded as positive, because throughout that region y is positive; and the area of the shaded part below the axis of x must be regarded as negative, because throughout that region y is negative. On adding these two areas together, we get zero.

8. Useful result.

To find the element of area enclosed between two radii vectores.

We require to find the area of POQ ($\equiv \delta A$). Let the circle, centre O and radius OP, cut OQ in P′.

Also let the circle, centre O and radius OQ, cut OP produced in Q′.

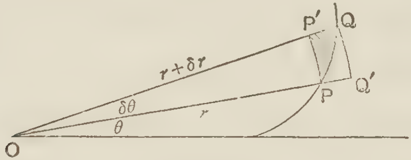

FIG. 158.

Then, from the figure, we have:

$$\text{Area of } P'PO < \text{area of } POQ < \text{area of } QQ'O.$$

$$\therefore\ \tfrac{1}{2}r^2\delta\theta < \delta A < \tfrac{1}{2}(r+\delta r)^2\delta\theta.$$

Hence, proceeding to the limit, we have $dA = \frac{1}{2}r^2 d\theta$.

9. To integrate a differential equation of the type $\frac{dy}{dx} = \frac{f(y)}{\phi(x)}$.

We write $\frac{dy}{f(y)} = \frac{dx}{\phi(x)}$ and integrate directly.

This process is called the process of "*integration by separation of the variables.*"

Example 1. *Integrate the differential equation* $\frac{dy}{dx} = x(y^2+1)$.

We have
$$\frac{dy}{y^2+1} = x\,dx.$$
$$\therefore\ \tan^{-1} y = \frac{x^2}{2} + C.$$

Example 2. *Integrate the differential equation* $x^2\frac{dy}{dx} = \frac{x^4+1}{y}$.

We have
$$y\,dy = \left(x^2 + \frac{1}{x^2}\right)dx.$$
$$\therefore\ \frac{y^2}{2} = \frac{x^3}{3} - \frac{1}{x} + C.$$

EXAMPLES XV.

1. Evaluate the following indefinite integrals :—

(i) $\int \sin^2 x\, dx$, (ii) $\int \sin^2 3x\, dx$, (iii) $\int \sin^2 (5x+7)\, dx$,

(iv) $\int \cos^2 x \sin x\, dx$, (v) $\int \cos^3 x \sin x\, dx$, (vi) $\int \sin^5 x \cos x\, dx$,

(vii) $\int \sec^2 3x\, dx$, (viii) $\int \tan^2 2x\, dx$, (ix) $\int x\sqrt{1-x^2}\, dx$,

(x) $\int x\sqrt{1+x^2}\, dx$, (xi) $\int x(a+bx^2)^{\frac{1}{2}}\, dx$, (xii) $\int x(1+x^2)^n\, dx$,

(xiii) $\int x(a+bx^2)^n\, dx$; (xiv) $\int \tan^3 x \sec^2 x\, dx$, (xv) $\int \tan^5 x \sec^2 x\, dx$,

(xvi) $\int \tan^n x \sec^2 x\, dx$, (xvii) $\int x \cos x^2\, dx$, (xviii) $\int x \sin x^2\, dx$,

(xix) $\int \frac{dx}{\sqrt{6x-x^2-8}}$, (xx) $\int \frac{dx}{1+4x^2}$, (xxi) $\int \frac{dx}{x^2+2x+2}$,

(xxii) $\int \frac{dx}{3x^2+6x+7}$, (xxiii) $\int \frac{dx}{\sqrt{25-9x^2}}$, (xxiv) $\int \frac{dx}{\sqrt{40-x^2-6x}}$,

(xxv) $\int \frac{dx}{x^2-x+1}$.

2. Evaluate the following definite integrals :—

(i) $\int_0^{\frac{\pi}{2}} \sin^2 x\, dx$, (ii) $\int_0^1 \sin^2 x\, dx$ (x in radians),

(iii) $\int_0^{0{\cdot}8} \cos^2 x\, dx$ (x in radians), (iv) $\int_{0{\cdot}2}^{0{\cdot}9} \cos^2 x\, dx$ (x in radians),

(v) $\int_0^{\frac{\pi}{2}} \cos^2 x \sin x\, dx$, (vi) $\int_0^{0{\cdot}8} \sin^2 x \cos x\, dx$ (x in radians),

(vii) $\int_0^{\frac{\pi}{4}} \tan^2 x\, dx$, (viii) $\int_0^{0{\cdot}7} x\sqrt{1-x^2}\, dx$, (ix) $\int_{0{\cdot}2}^{0{\cdot}9} x\sqrt{1+x^2}\, dx$,

(x) $\int_0^3 \left(\frac{5}{3}x+\frac{4}{9}\right)^{\frac{1}{2}} dx$, (xi) $\int_0^3 \left(\frac{1}{3}x+\frac{9}{16}\right)^{-\frac{1}{2}} dx$, (xii) $\int_5^{5\frac{1}{2}} \frac{dx}{\sqrt{10x-x^2-24}}$,

(xiii) $\int_0^{\frac{1}{3}} \frac{dx}{1+9x^2}$, (xiv) $\int_0^1 \frac{dx}{x^2+4x+5}$, (xv) $\int_{\frac{7}{2}}^7 \frac{dx}{(49-x^2)^{\frac{1}{2}}}$,

(xvi) $\int_{\frac{1}{2}}^3 \frac{dx}{\sqrt{21-x^2-4x}}$.

3. Evaluate the following indefinite integrals :—

(i) $\int \frac{x\,dx}{\sqrt{1-x^4}}$, (ii) $\int \frac{x\,dx}{1+x^4}$, (iii) $\int \frac{x^2\,dx}{\sqrt{9-x^6}}$, (iv) $\int \frac{x^2\,dx}{25+x^6}$,

(v) $\int \cos^3 x\,dx$, (vi) $\int \cos^5 x\,dx$, (vii) $\int \sin^5 x\,dx$, (viii) $\int x(1-x)^{\frac{3}{2}}\,dx$,

(ix) $\int x(1-x)^{-\frac{1}{2}}\,dx$, (x) $\int x^2\sqrt{1+x}\,dx$, (xi) $\int x^5\sqrt{1+x^2}\,dx$,

(xii) $\int \sin\theta(1-e^2\cos^2\theta)^{-\frac{1}{2}}\,d\theta$, (xiii) $\int \cos\theta(1-e^2\sin^2\theta)^{-\frac{1}{2}}\,d\theta$,

(xiv) $\int \frac{dx}{x\sqrt{x^2-1}}$ $\left[\textit{Hint.}\text{—Put } x=\frac{1}{y}\right]$, (xv) $\int \frac{dx}{x\sqrt{a^2x^2-1}}$.

4. Integrate (i) $\int \sqrt{9-x^2}\,dx$ [*Hint.*—Put $x=3\sin\theta$],

(ii) $\int \sqrt{25-x^2}\,dx$, (iii) $\int \sqrt{a^2-x^2}\,dx$,

(iv) $\int \sqrt{1-e^2x^2}\,dx$, (v) $\int \sin\theta(1-e^2\cos^2\theta)^{\frac{1}{2}}\,d\theta$.

(vi) $\int \cos\theta(1-e^2\sin^2\theta)^{\frac{1}{2}}\,d\theta$, (vii) $\int \sec^2\theta(1-e^2\tan^2\theta)^{\frac{1}{2}}\,d\theta$.

5. Integrate (i) $\int \frac{dx}{x^2(1+x^2)}$ $\left[\textit{Hint.}\text{—}\frac{1}{x^2(1+x^2)}=\frac{1}{x^2}-\frac{1}{1+x^2}\right]$,

(ii) $\int \frac{dx}{(1+2x^2)(1+3x^2)}$, (iii) $\int \frac{dx}{3+31x^2+36x^4}$, (iv) $\int \frac{1+x^{-2}}{1+x^2}\,dx$.

6. Integrate (i) $\int \sqrt{\frac{1+x}{1-x}}\,dx$ [*Hint.*—Put $x=\cos 2\theta$],

(ii) $\int \sqrt{\frac{1-x}{1+x}}\,dx$, (iii) $\int \sqrt{\frac{x-1}{x+1}}\,\frac{dx}{x^2}$ $\left[\textit{Hint.}\text{—Put } x=\frac{1}{y}\right]$,

(iv) $\int \sqrt{\frac{x+1}{x-1}}\,\frac{dx}{x^2}$, (v) $\int \sqrt{\frac{x}{1-x}}\,dx$ [*Hint.*—Put $x=\sin^2\theta$],

(vi) $\int \frac{x}{\sqrt{x-1}}\,dx$ [*Hint.*—Put $x=\sec^2\theta$].

7. Evaluate

(i) $\int_0^3 \sqrt{9-x^2}\,dx$, (ii) $\int_0^7 \sqrt{49-x^2}\,dx$, (iii) $\int_0^2 \sqrt{9-x^2}\,dx$ (2 places of dec.),

(iv) $\int_0^1 \sqrt{9-x^2}\,dx$ (2 places of dec.), (v) $\int_0^4 \sqrt{25-x^2}\,dx$ (2 places of dec.),

vi) $\int_2^7 \frac{dx}{x\sqrt{x^2-1}}$ (2 places of dec.).

8. Evaluate correct to 2 places of dec.

(i) $\int_0^{0{\cdot}9} \sqrt{1-x}\,dx$, (ii) $\int_1^5 \frac{dx}{x^2(1+x^2)}$, (iii) $\int_0^{0{\cdot}8} \sqrt{\frac{1+x}{1-x}}\,dx$,

(iv) $\int_{0{\cdot}2}^{0{\cdot}9} \sqrt{\frac{1+x}{1-x}}\,dx$, (v) $\int_0^{0{\cdot}7} \sqrt{\frac{1-x}{1+x}}\,dx$, (vi) $\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$,

(vii) $\int_0^1 \frac{dx}{\sqrt{5-2x^2}}$, (viii) $\int_1^2 \frac{dx}{\sqrt{18-3x^2}}$.

9. Integrate by parts

(i) $\int x\sin x\,dx$, (ii) $\int x\cos x\,dx$, (iii) $\int x(3\sin x+4\cos x)\,dx$,

(iv) $\int x(1+x)^{20}\,dx$, (v) $\int x(1+x)^{49}\,dx$, (vi) $\int x(1+x)^{n+1}\,dx$,

(vii) $\int x\sin 2x\,dx$, (viii) $\int x\cos 3x\,dx$, (ix) $\int \sin^{-1}x\,dx$,

(x) $\int \cos^{-1}x\,dx$, (xi) $\int x\tan^{-1}x\,dx$, (xii) $\int \frac{\tan^{-1}x}{x^3}\,dx$,

(xiii) $\int x^2\cos x\,dx$, (xiv) $\int x^2\sin x\,dx$.

10. Evaluate

(i) $\int_0^{\frac{\pi}{4}} x\cos x\,dx$, (ii) $\int_0^{0{\cdot}8} x\cos x\,dx$ (x in radians), (iii) $\int_0^{\frac{\pi}{2}} x(\sin x+\cos x)\,dx$,

(iv) $\dfrac{\int_0^{\frac{\pi}{2}} x\cos x\,dx}{\int_0^{\frac{\pi}{2}} x\sin x\,dx}$, (v) $\int_0^{\frac{\pi}{2}} x\sin^2 x\,dx$, (vi) $\int_0^{\frac{\pi}{2}} x\cos^2 x\,dx$.

11. Transform $\int_0^{\frac{\pi}{2}} x \sin^2 x\, dx$ by the substitution $x = \frac{\pi}{2} - y$, and state any noticeable result thus obtained.

12. AB and CD are two parallel chords of a circle of radius 5 cm. subtending angles of 60° and 30° respectively at the centre and on the same side of it. Calculate the area of the circle intercepted between the two chords.

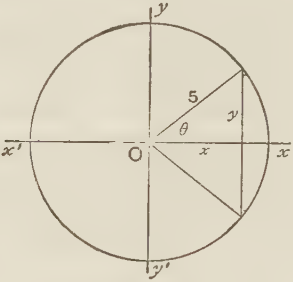

FIG. 159.

[*Hint.*—Area $= 2\int y\, dx$, where $x = 5 \cos \theta$ and $y = 5 \sin \theta$.]

13. Find the ratio of the areas of the two parts of a circle of radius 1 cm. in which it is divided by a chord subtending an angle of 42° at the centre.

14. AB and CD are two parallel chords of a circle of radius a cm. subtending angles of 2α and 2β radians respectively at the centre. Calculate the area of the circle intercepted between the two chords.

15. Find the ratio of the areas of the two parts of a circle of radius a cm. in which it is divided by a chord subtending an angle 2α radians at the centre.

16. A plane subtends at the centre of a sphere of radius 1 ft. a cone whose semi-vertical angle is 30°. Find the ratio of the volumes of the two parts of the sphere into which it is divided by the plane.

17. A plane subtends at the centre of a sphere of radius a yd. a cone whose semi-vertical angle is α radians. Find the ratio of the volumes of the two parts of the sphere into which it is divided by the plane.

18. Trace the cardioid whose equation is $r = a(1 + \cos \theta)$, and find the area enclosed by it.

19. Trace the lemniscate whose equation is $r^2=a^2 \cos 2\theta$, and find the area enclosed by one of its loops.

20. Prove that in the case of the hyperbolic spiral whose equation is $r\theta=a$ the area bounded by an arc of the curve and the radii vectores of the ends of the arc is proportional to the difference of the lengths of these radii vectores.

21. Trace the curve whose equation is $r=a \sin 3\theta$, and find the area enclosed by one of its loops.

22. Trace the curve whose equation is $r=a \sin 5\theta$, and find the area enclosed by one of its loops.

23. Find the area enclosed by the ellipse whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. [*Hint.*—Let $x=a \sin \phi$, then $y=b \cos \phi$.]

24. Any point on the cycloid is given by the equations $x=a(\theta+\sin \theta)$, $y=a(1+\cos \theta)$, the axis of x being the straight line on which the generating circle rolls. Find the area enclosed between this line and one loop of the curve.

25. Trace the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, and find the area enclosed by it. [*Hint.*—Let $x=a \sin^3 \phi$, then $y=a \cos^3 \phi$.]

26. Find the volume of a cap of a sphere of radius 10 in., the cone which has its vertex at the centre of the sphere and stands on the cap as base having a semi-vertical angle of 40°.

27. Find the volume of the solid generated by the revolution of one loop of the curve $y=\sin x$ (x in radians) about the axis of x.

28. Find the volume generated by the revolution about the axis of x of one loop of the cycloid of question 24.

29. Find the position of the Centre of Gravity of a semicircle radius a.

30. Find the position of the Centre of Gravity of one of the halves into which the axis of y cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

31. Find the position of the Centre of Gravity of one of the halves into which the axis of x cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

32. Find the position of the Centre of Gravity of a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

33. Find the position of the Centre of Gravity of a segment of a circle of radius a cm., the height of the segment being h cm.

34. Find the position of the Centre of Gravity of a segment of a circle of radius a in., the arc of the segment subtending an angle 2α radians at the centre of the circle.

35. Find the position of the Centre of Gravity of a hemisphere of radius a ft.

36. Find the position of the Centre of Gravity of a cap of a sphere of radius a cm., the height of the cap being h cm.

37. Find the M.I. of a circular disc radius a ft. and mass M lb. about one of its diameters.

38. Find the M.I. of an elliptical disc whose semi-major and semi-minor axes are a cm. and b cm. and mass M grams about its major axis.

39. Find the M.I. of the elliptical disc of the preceding question about its minor axis.

40. Find the M.I. of the elliptical disc of the two preceding questions about an axis through its centre perpendicular to its plane.

CHAPTER XVI.

LENGTHS OF CURVES AND AREAS OF SURFACES.

1. Practical calculation of length of curve.

Consider the subjoined graph, in which we wish to obtain approximately the length of the arc of the curve.

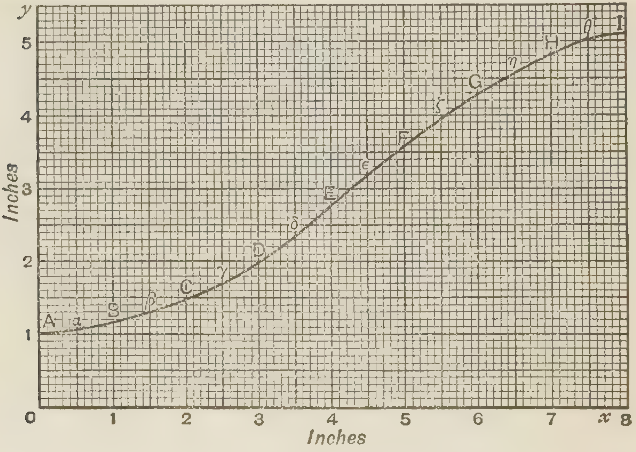

FIG. 160.

We can proceed in either of two ways: (1) by measuring the curve with a piece of string, as described in text-books on Practical Physics; (2) by performing certain calculations.

We shall deal with each of these methods separately.

METHOD (1). *By Measurement with a String.*

We shall quote the description of this method given in Rintoul's *Introduction to Practical Physics.*

"A curved line may be measured by means of a flexible cord, such as a piece of cotton, in the following manner:—Suppose that we have to measure the length of the curve in the subjoined figure (fig. 161). Take a piece of cotton of sufficient length, and tie a knot near one end. Place this knot on the paper at A, and hold it tight with the left hand. With the other hand stretch the cotton along the curve to such a distance AX that the cotton coincides with the curve. (It is obvious that this distance AX will depend on the curvature—the greater the curvature the shorter the length of cotton which will coincide with the curve.) Hold the cotton down on the curve at X with the thumb of the right hand, let go with the left hand, and place the thumb or forefinger of the left hand close to the right at X. Holding the thread at X with the left hand, stretch the cotton another short distance XY as before. In this way the cotton can be laid along the whole of the curve ABC. When C is reached with the thumb of the right hand, lift the cotton with this hand without allowing the cotton to slip, and lay it straight along the metre measure. Thus the length of the curve can be measured by measuring the length of the cotton which can be made to lie along it."

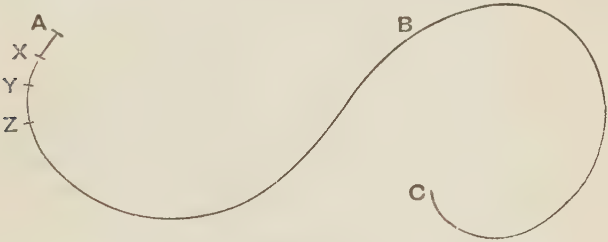

FIG. 161.

If we carry out these instructions with regard to the above curve drawn on the squared paper and take the average of five readings, we find that the length of the curve is 9·08 in. approximately.

What has actually happened in the foregoing process is that we have taken the measure of the inscribed polygon consisting of the straight lines (or chords) AX, XY, YZ, etc. It is plain that the shorter the arcs AX, XY, YZ, etc., the more accurate will the above measurement be. But the above method can never (quite apart from practical disabilities) give us a perfectly accurate result.

METHOD (2). *By Graphical Calculation.*

In graphical calculation we proceed as follows:—

Let, in fig. 162, PQ be a portion of the arc of a curve which—for practical approximative purposes — may be considered as a straight line. (In other words, if the "stretched string" method were being used, the string, if stretched between P and Q, would to the naked eye appear to coincide with the arc PQ.)

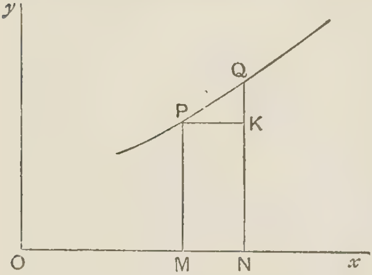

FIG. 162.

Now $PQ^2 = PK^2 + KQ^2$.

$$\therefore\ PQ = \sqrt{PK^2 + KQ^2}$$

$$= \sqrt{MN^2 + (\text{difference of ordinates of Q and P})^2}.$$

Let us now apply this method to the curve in fig. 160. Let us consider the series of points A, B, C, D, . . . , I.

We read off and tabulate as follows:—

Ordinate of A = 1·00 inches
,, B = 1·15 ,,
,, C = 1·47 ,,
,, D = 1·97 ,,
,, E = 2·74 ,,
,, F = 3·51 ,,
,, G = 4·30 ,,
,, H = 4·82 ,,
,, I = 5·10 ,,

Hence, applying the method of fig. 162, we get:—

$$AB^2 = 1^2 + (1{\cdot}15 - 1{\cdot}00)^2 = 1{\cdot}0225$$
$$BC^2 = 1^2 + (1{\cdot}47 - 1{\cdot}15)^2 = 1{\cdot}1024$$
$$CD^2 = 1^2 + (1{\cdot}97 - 1{\cdot}47)^2 = 1{\cdot}2500$$
$$DE^2 = 1^2 + (2{\cdot}74 - 1{\cdot}97)^2 = 1{\cdot}5929$$
$$EF^2 = 1^2 + (3{\cdot}51 - 2{\cdot}74)^2 = 1{\cdot}5929$$
$$FG^2 = 1^2 + (4{\cdot}30 - 3{\cdot}51)^2 = 1{\cdot}6241$$
$$GH^2 = 1^2 + (4{\cdot}82 - 4{\cdot}30)^2 = 1{\cdot}2704$$
$$HI^2 = 1^2 + (5{\cdot}10 - 4{\cdot}82)^2 = 1{\cdot}0784$$

Hence AB = 1·01 inches
BC = 1·05 „
CD = 1·12 „
DE = 1·26 „
EF = 1·26 „
FG = 1·27 „
GH = 1·13 „
HI = 1·04 „

Hence the total length of the curve is 9·1 inches approximately, which agrees very closely with the result given by the method of the stretched string.

2. Exact evaluation of the length of a curve.

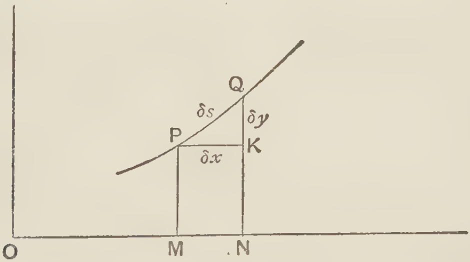

FIG. 163.

The above figure explains itself, except that δs is usually taken to denote the length of a short arc of the curve.

Then plainly,

$$\text{the arc PQ} \equiv \{(\delta x)^2 + (\delta y)^2\}^{\frac{1}{2}} \text{ (very approx.)}$$

Hence, approximately, the length of the total curve

$$\equiv \sum \{(\delta x)^2 + (\delta y)^2\}^{\frac{1}{2}} \text{ (very approx.)}$$

$$\equiv \sum \left\{1 + \left(\frac{\delta y}{\delta x}\right)^2\right\}^{\frac{1}{2}} \delta x \text{ (very approx.)}$$

$$\therefore \text{ the exact length of the total curve } \equiv \int \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx.$$

Example. *Sketch the graph of $y^2 = x^3$, and hence find by integration the length of the arc from $x = 0$ to $x = 10$.*

$$\because y^2 = x^3, \qquad \therefore y = x^{\frac{3}{2}}.$$

$$\therefore \frac{dy}{dx} = \frac{3}{2} x^{\frac{1}{2}}.$$

$$\therefore s = \int_0^{10} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \int_0^{10} \sqrt{1 + \frac{9}{4}x}\, dx$$

$$= \frac{4}{9} \int_0 \left(1 + \frac{9}{4}x\right)^{\frac{1}{2}} d\left(1 + \frac{9}{4}x\right)$$

$$= \frac{4}{9} \times \frac{2}{3} \times \left[\left(1 + \frac{9}{4}x\right)^{\frac{3}{2}}\right]_0^{10}$$

$$= 33{\cdot}5.$$

3. Area of a surface of revolution.

Consider a curve AB and a line OX both lying in the same plane. Then if we imagine this plane to rotate about OX as

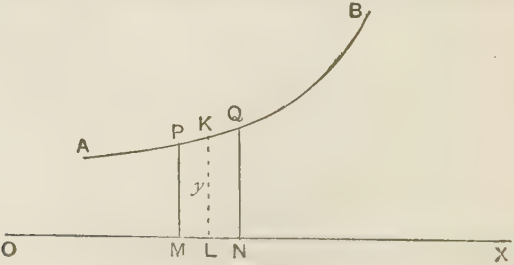

FIG. 164.

an axis carrying the curve AB along with it, the rotating curve AB will trace out a surface of revolution having OX as its "line of symmetry" or axis.

To obtain the area of a surface of revolution, we shall begin as usual by considering how we may find approximately the area of a surface of revolution traced out by a curve drawn on squared paper.

Let us first of all consider the arc AB in fig. 165. The point a in its revolution about Ox describes a circle of radius an.

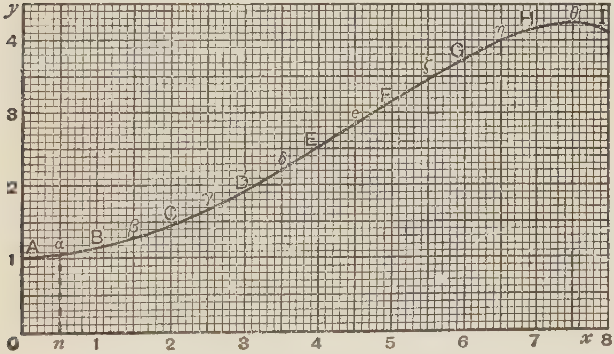

FIG. 165.

Hence the distance described by a during the revolution of the curve is $2\pi . an$.

Hence the area of the portion of surface traced out by the arc AB in its revolution about Ox will be very approximately $2\pi an$. AB.

If we therefore treat each portion of arc, AB, BC, CD, etc. separately, and sum all the portions of surface thus obtained, we shall obtain very approximately the total area of the required surface of revolution.

Let us, therefore, find in fig. 165 the lengths of the arcs AB, BC, CD, etc., either by the "stretched string" or by the "graphical calculation" method, and let us tabulate our results as follows :—

AB = 1·01 in.; ordinate of $\alpha = 1{\cdot}06$ in.
BC = 1·05 „ ; „ $\beta = 1{\cdot}30$ „
CD = 1·09 „ ; „ $\gamma = 1{\cdot}68$ „
DE = 1·18 „ ; „ $\delta = 2{\cdot}20$ „
EF = 1·19 „ ; „ $\epsilon = 2{\cdot}84$ „
FG = 1·17 „ ; „ $\zeta = 3{\cdot}48$ „
GH = 1·09 „ ; „ $\eta = 4{\cdot}00$ „
HI = 1·07 „ ; „ $\theta = 4{\cdot}24$ „

Hence we obtain :—

Portion of surface traced out by AB = 6·73 sq. in.
„ „ „ „ BC = 8·58 „
„ „ „ „ CD = 11·50 „
„ „ „ „ DE = 16·31 „
„ „ „ „ EF = 21·23 „
„ „ „ „ FG = 25·59 „
„ „ „ „ GH = 27·39 „
„ „ „ „ HI = 28·51 „

∴ total area of surface = 145·84 sq. in. (very approx.).

4. Exact evaluation of the area of a surface of revolution.

Let $PQ \equiv \delta s$ be a very short arc of the given curve. Let y be the ordinate of G, the mid-point of the arc PQ. Then, as the arc rotates about Ox, the point G traverses a path of length $2\pi y$. The portion of surface traced out by the arc $PQ \equiv \delta s$ is therefore $2\pi y\,\delta s$ very approximately.

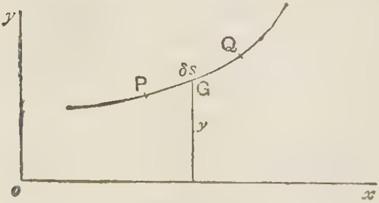

FIG. 166.

Hence total area of surface

$$= \sum 2\pi y\,\delta s \text{ (very approx.)}$$

$$= 2\pi \sum y\,\delta s \text{ (very approx.)}.$$

Hence, in the limit, the exact value for the area of the surface of revolution is $2\pi \int y\,ds$.

N.B. Mistakes are frequently made by taking the element of surface of revolution to be $2\pi y\,\delta x$ instead of $2\pi y\,\delta s$. The student may avoid this error once for all if he notes that the surface of revolution is generated by the arc δs *and not by* δx.

Corollary. GULDINUS AND PAPPUS' THEOREM.—*If the arc of a plane curve revolve about an axis lying in its plane but not intersecting it, the area of the surface of revolution generated by the arc is equal to the length of the arc, multiplied by the distance traversed by the centre of gravity of the arc during the revolution.*

Example 1. *The parabola* $y^2=16x$ *is rotated about its axis. Find the area of the surface of revolution thus generated, which is intercepted between the vertex and a plane perpendicular to the axis, and at a distance* $x=10$ *from the vertex.*

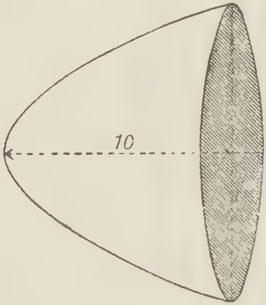

FIG. 167.

$$\text{Area of surface}=2\pi\int y\,ds$$

$$=2\pi\int_0^{10} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx. \quad\text{.......................}(1)$$

But $$y=4x^{\frac{1}{2}}.$$

$$\therefore\ \frac{dy}{dx}=2x^{-\frac{1}{2}}. \quad\text{..}(2)$$

$$\therefore\ \text{Area}=2\pi\int_0^{10}4x^{\frac{1}{2}}\cdot\sqrt{1+\frac{4}{x}}\,dx \quad \text{by (1) and (2)}$$

$$=8\pi\int_0^{10}(4+x)^{\frac{1}{2}}\,dx$$

$$=743{\cdot}5.$$

Example 2. *To prove that the area of the surface of a sphere intercepted between two parallel planes is equal to the area intercepted between the parallel planes of the part of the surface of the circumscribing cylinder whose generating lines are perpendicular to the parallel planes.*

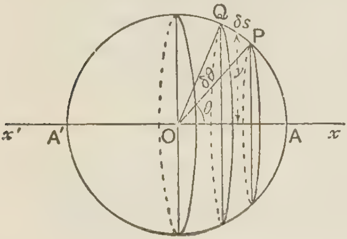

FIG. 168.

Consider the area of the portion of surface generated by the small arc PQ, and let the semi-angles subtended by the given parallel planes at the centre of the sphere be α and β respectively.

Then area of surface required

$$=2\pi\int y\,ds$$

$$=2\pi\int_\alpha^\beta a\sin\theta\times a\,d\theta\ (\text{since } \delta s=a\,\delta\theta)$$

$$=2\pi a^2\int_\alpha^\beta \sin\theta\,d\theta$$

$$=2\pi a^2\Big[-\cos\theta\Big]_\alpha^\beta$$

$$=2\pi a^2(\cos\alpha-\cos\beta)$$

$$=2\pi ac$$

(where c is the perpendicular distance between the given parallel planes)

=area of portion of circumscribing cylinder intercepted between the given planes (the generators of the cylinder being parallel to Ox).

Example 3. *Evaluate* $2\pi\int_{t=0}^{t=4} y\,ds$, *given that* $x=at^2$, $y=2at$.

We have

$$dx=2at\,dt,$$

$$dy=2a\,dt.$$

$$\therefore\ (ds)^2=(dx)^2+(dy)^2$$

$$=4a^2t^2(dt)^2+(4a^2)(dt)^2$$

$$=4a^2(t^2+1)(dt)^2.$$

$$\therefore\ ds=2a\sqrt{1+t^2}\,dt.$$

$$\therefore\ 2\pi\int_{t=0}^{t=4} y\,ds = 2\pi\int_0^4 2at\,.\,2a\sqrt{1+t^2}\,dt$$

$$= 8\pi a^2\int_0^4 t\sqrt{1+t^2}\,dt$$

$$= 8\pi a^2\Big[\tfrac{1}{3}(1+t^2)^{\frac{3}{2}}\Big]_0^4$$

$$= \frac{8\pi}{3}\,a^2(17\sqrt{17}-1).$$

5. Given that the surface of a sphere of radius r is $4\pi r^2$, to find the volume of a sphere of radius a.

Consider the volume δV bounded between two concentric spheres of radii r and $r+\delta r$ respectively.

If we imagine the substance intercepted between the two spherical surfaces to be made of very flexible india-rubber, we can suppose a cut to be made and the india-rubber to be laid out flat on a table without the inner surface of the rubber being either compressed or extended. Plainly, what was the outer surface will be compressed, and we will get a flat pair of equal surfaces of area $4\pi r^2$, whose distance apart is δr.

$$\therefore\ 4\pi r^2\,\delta r < \delta V$$

(since compression of the volume has taken place).

In the same way, if we imagined the outer surface, whose area is $4\pi(r+\delta r)^2$, to be flattened out on the table, what was the inner surface would be stretched out from $4\pi r^2$ to $4\pi(r+\delta r)^2$, and the volume would be increased from δV to $4\pi(r+\delta r)^2\,\delta r$.

$$\therefore\ 4\pi r^2\,\delta r < \delta V < 4\pi(r+\delta r)^2\,\delta r.$$

Hence, proceeding to the limit, we obtain $dV = 4\pi r^2\,dr$.

$$\therefore\ \text{Volume} = \int_{r=0}^{r=a} dV = \int_0^a 4\pi r^2\,dr$$

$$= \frac{4}{3}\pi a^3.$$

6. Integration by the Method of Reduction.

It is frequently convenient, when an expression to be integrated is complicated, to express the given integral as the sum of an integrated expression plus the integral of a less complicated expression. We can illustrate by means of an example.

Consider the following :—

$$\frac{d}{dx}\left(x^n \sin x\right) = x^n \cos x + nx^{n-1} \sin x.$$

$$\therefore \int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx.$$

Hence $\int x^n \cos x \, dx$ has been expressed in terms of an integrated expression plus the integral of an expression of lower degree in x. We can then proceed to "reduce" $\int x^{n-1} \sin x \, dx$ in a similar way, and hence finally the required integration will be effected.

Example 1. *Evaluate* $\int x^3 \cos x \, dx$.

Let us differentiate some expression which will contain $x^3 \cos x$ in the answer. Obviously, by inspection, $x^3 \sin x$ is such an expression.

We have $$\frac{d}{dx}(x^3 \sin x) = x^3 \cos x + 3x^2 \sin x.$$

$$\therefore \int x^3 \cos x \, dx = x^3 \sin x - 3\int x^2 \sin x \, dx \quad \text{.................(1)}$$

Let us next differentiate some expression which contains $x^2 \sin x$ in the answer. Obviously $x^2 \cos x$.

We have $$\frac{d}{dx}(x^2 \cos x) = -x^2 \sin x + 2x \cos x.$$

$$\therefore \int x^2 \sin x \, dx = -x^2 \cos x + 2\int x \cos x \, dx \quad \text{..............(2)}$$

Finally, an expression which on being differentiated will contain $x \cos x$ is $x \sin x$.

Hence $$\frac{d}{dx}(x \sin x) = x \cos x + \sin x.$$

$$\therefore \int x \cos x \, dx = x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x. \quad \text{.........................(3)}$$

Hence, by (1), (2), and (3),

$$\int x^3 \cos x\, dx = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x$$
$$= (x^3 - 6x) \sin x + (3x^2 - 6) \cos x.$$

Example 2. *Evaluate* $\int \sin^4 x\, dx$.

An expression which on being differentiated will contain $\sin^4 x$ is plainly $\sin^3 x \cos x$.

$$\therefore \frac{d}{dx}(\sin^3 x \cos x) = -\sin^4 x + 3 \sin^2 x \cos^2 x$$
$$= -\sin^4 x + 3 \sin^2 x(1 - \sin^2 x)$$
$$= -4 \sin^4 x + 3 \sin^2 x.$$

$$\therefore \int \sin^4 x\, dx = \tfrac{3}{4}\int \sin^2 x\, dx - \tfrac{1}{4} \sin^3 x \cos x.$$

Hence $\int \sin^4 x\, dx$ is reduced to depend on $\int \sin^2 x\, dx$.

Otherwise, $\int \sin^4 x\, dx = -\int \sin^3 x\, d \cos x$

$$= -\sin^3 x \cos x + \int \cos x\, d \sin^3 x$$
$$= -\sin^3 x \cos x + 3\int \sin^2 x \cos^2 x\, dx$$
$$= -\sin^3 x \cos x + 3\int (\sin^2 x - \sin^4 x) dx.$$

$$\therefore 4\int \sin^4 x\, dx = -\sin^3 x \cos x + 3\int \sin^2 x\, dx.$$

Hence $\int \sin^4 x\, dx = \tfrac{3}{4}\int \sin^2 x\, dx - \tfrac{1}{4} \sin^3 x \cos x.$

Example 3. *Evaluate* $\int \sqrt{1 - x^2}\, dx$.

Integrating by parts, we have

$$\int \sqrt{1 - x^2}\, dx = x\sqrt{1 - x^2} - \int x\, d\sqrt{1 - x^2}$$
$$= x\sqrt{1 - x^2} + \int \frac{x^2\, dx}{\sqrt{1 - x^2}} \quad \text{......................}(1)$$

Also
$$\int \sqrt{1-x^2}\,dx = \int \frac{1-x^2}{\sqrt{1-x^2}}\,dx$$
$$= \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x^2\,dx}{\sqrt{1-x^2}}$$
$$= \sin^{-1} x - \int \frac{x^2\,dx}{\sqrt{1-x^2}} \quad \text{.....................(2)}$$

Hence, by adding (1) and (2) and dividing by 2,
$$\int \sqrt{1-x^2}\,dx = \tfrac{1}{2}x\sqrt{1-x^2} + \tfrac{1}{2}\sin^{-1} x.$$

EXAMPLES XVI.

1. Trace the graph of $y = \frac{x^2}{3}$ from $x = -5$ to $x = +5$ as accurately as you can.

(i) Obtain the length of the arc by the "stretched string" method.
(ii) Obtain the length of the arc by "graphical calculation."
(iii) State the percentage difference in your two answers to (i) and (ii).

2. Trace the graph of $y = \frac{x^2+x}{4}$ from $x = -5$ to $x = +5$ as accurately as you can, and answer the same three questions as in question 1.

3. Find by integration the length of the arc of the curve $y^2 = x^3$ from $x = 0$ to $x = 5$.

4. Find the length of the curve $y^2 = x^3$ from $x = 5$ to $x = 12$.

5. Find the length of the arc of the curve $y^2 = 4x^3$ from $x = 0$ to $x = 10$.

6. Find the length of the arc of the curve $y^2 = 3{\cdot}7x^3$ from $x = 0$ to $x = 10$.

7. Find the length of the arc of the curve $y^2 = a^2x^3$ from $x = 0$ to $x = 1$.

8. Find the length of the arc of the curve $x^3 = a^3y^2$ from $x = 0$ to $x = a$.

9. Find by integration the surface of a right circular cone of semi-vertical angle 23° and of height 12 cm.

10. Find by integration the surface of a right circular cone of semi-vertical angle 19° and of height 20 cm.

11. The diameters of the cross-sections of a frustum of a cone are 5 cm. and 8 cm. respectively, and the distance between them is 9 cm. Calculate the total surface of the frustum (using integration).

12. A round chimney-stack stands 30 ft. high. The diameter of its base is 6 ft. and of its top 1 ft. Calculate the total surface of the chimney.

13. The figure represents a brick funnel of height 6 ft., and the external and internal diameters of the upper section are 6 ft. and 5 ft. respectively. The corresponding diameters of the lower section are 2 ft. and 1 ft. respectively. Find by integration the difference between the outer and inner sloping surfaces.

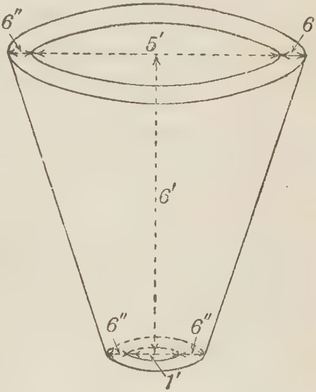

Fig. 1.9.

14. Find the area of zinc required to construct an ordinary bucket with a bottom and lid, given that the diameter of the lid is $1\frac{1}{2}$ ft., the diameter of the bottom 1 ft., and the height of the bucket 2 ft.

15. Find the area of the surface of a sphere of radius 5 ft. intercepted between a diametral plane and a parallel plane $2\frac{1}{2}$ ft. distant.

16. Find the area of the surface of a sphere of radius 12 ft. intercepted between two parallel planes at distances of 2 ft. and 10 ft. respectively from the centre.

17. An upturned granite hemisphere of radius 7 inches is placed in the basin of a fountain in which the water is 5 inches deep. Calculate the wetted area of the surface of the hemisphere.

18. Find how deep the water in an upturned hemispherical pot of radius 2 ft. must be in order that the surface may be half-wetted.

19. Trace roughly the curve whose equation is $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ and find

(i) the length of its perimeter.

(ii) the area of the surface formed by the revolution of the curve about the axis of x.

20. AB is the diameter of a sphere and N is a point in AB. Prove that the plane through N perpendicular to AB divides the surface of the sphere into two parts whose ratio is AN : NB.

21. The parabola $y^2=4ax$ is rotated about its axis. Prove that the area of the surface intercepted between the vertex and $x=c$ is

$$\frac{8\pi}{3}\sqrt{a}\,\{(c+a)^{\frac{3}{2}}-a^{\frac{3}{2}}\}.$$

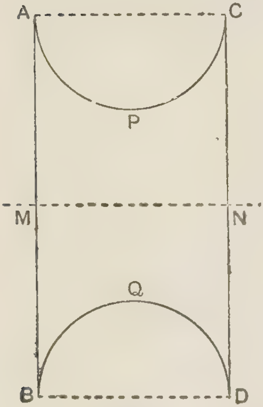

FIG. 170.

22. Fig. 170 is a section of a napkin-ring by a plane through the axis MN where APC and BQD are semicircles. (The napkin-ring may be supposed generated by the revolution of APC about the axis MN). If the napkin-ring be 1 inch wide (*i.e.* AC=1 inch) and 2 inches in diameter (*i.e.* AB=2 inches), calculate the surface of the napkin-ring.

23. An anchor-ring may be supposed generated by the revolution of the circle in fig. 171 about the axis XY. O is the centre of the circle

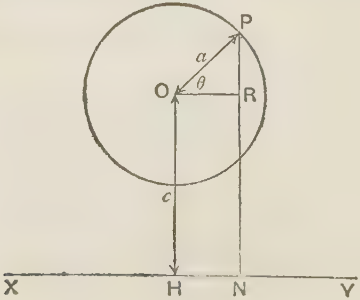

FIG. 171.

Given that $OH=c$, $OP=a$, $PN=y$, $P\hat{O}R=\theta$, $HN=x$,

(i) Find x in terms a and θ.
(ii) Find y in terms of a, c, θ.
(iii) Find dx in terms of a, θ, $d\theta$.
(iv) Find dy in terms of a, c, θ, $d\theta$.
(v) Calculate $(dx)^2+(dy)^2$ in terms of a, c, θ, $d\theta$.
(vi) Calculate ds in terms of a, c, θ, $d\theta$.
(vii) Write down an expression as an integral for the surface area of the anchor-ring in terms of a and c.
(viii) Find the surface-area of the anchor-ring.
(ix) Explain a simpler geometrical way of obtaining ds in terms of $d\theta$ than that outlined in (iii), (iv), (v), (vi).

24. Obtain the volume of the anchor-ring in question 23.

25. The diameter of the aperture of an anchor-ring is 2 ft. The radius of the generating circle is half a foot. Calculate the volume and surface (using integration).

26. An anchor-ring is laid down on a table and a plane is taken parallel to the table and at a distance x ft. from the centre of the anchor-ring. The radius of the generating circle is a ft., and the distance of the centre of the generating circle from the centre of the anchor-ring is c ft.

(i) Prove that the area of the section of the anchor-ring by the above plane is

$$4\pi c\sqrt{a^2-x^2}.$$

(ii) Hence show that the volume of the anchor-ring is

$$\int_{-a}^{+a} 4\pi c\sqrt{a^2-x^2}\,dx.$$

(iii) Evaluate the integral in (ii).

27. An anchor-ring whose generating circle is 1 ft. in diameter, and the diameter of whose aperture is 2 ft., lies on the deck of a ship in 4 in. of water. Calculate the area of the surface wetted.

28. A spherical dome stands on a circular base of diameter 10 ft. The height of the dome is 3 ft. Find the cost of covering its surface with sheet lead at 4½d. per sq. ft.

29. Find the length of the arc of the curve

$$x=a(\theta+\sin\theta)$$
$$y=a(1+\cos\theta$$

from $\theta=0$ to $\theta=\frac{\pi}{2}$.

30. Find the area of the surface generated by the revolution of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about (i) the axis of x, (ii) the axis of y.

31. Find the position of the centre of gravity of an arc of a circle of radius a which subtends an angle of 2α radians at the centre of the circle. Hence deduce the position of the centre of gravity of a semicircular arc.

32. Find the position of the centre of gravity of the surface of a sphere of radius a which subtends at the centre of the sphere the solid angle of a cone of semi-vertical angle α radians. Hence deduce the position of the centre of gravity of the surface of a hemisphere.

33. Evaluate (i) $\int x^4 \cos x\, dx$, (ii) $\int x^5 \sin x\, dx$.

34. Prove the formulae $\int \cos^n \theta\, d\theta = \frac{\sin\theta \cos^{n-1}\theta}{n} + \frac{n-1}{n}\int \cos^{n-2}\theta\, d\theta.$

$$\int \sin^n \theta\, d\theta = -\frac{\cos\theta \sin^{n-1}\theta}{n} + \frac{n-1}{n}\int \sin^{n-2}\theta\, d\theta.$$

35. Find the value of

(i) $\int_0^{\frac{\pi}{2}} \cos^3\theta\, d\theta$, (ii) $\int_0^{\frac{\pi}{2}} \cos^5\theta\, d\theta$, (iii) $\int_0^{\frac{\pi}{2}} \cos^7\theta\, d\theta$, (iv) $\int_0^{\frac{\pi}{2}} \cos^9\theta\, d\theta$.

36. Find the value of

(i) $\int^{\frac{\pi}{2}} \cos^4\theta\, d\theta$, (ii) $\int_0^{\frac{\pi}{2}} \cos^6\theta\, d\theta$, (iii) $\int_0^{\frac{\pi}{2}} \cos^8\theta\, d\theta$, (iv) $\int_0^{\frac{\pi}{2}} \cos^{10}\theta\, d\theta$.

37. Find the value of

(i) $\int_0^{\frac{\pi}{2}} \sin^3\theta\, d\theta$, (ii) $\int_0^{\frac{\pi}{2}} \sin^5\theta\, d\theta$, (iii) $\int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta$, (iv) $\int_0^{\frac{\pi}{2}} \sin^8\theta\, d\theta$.

38. If m is a positive integer, find the value of

(i) $\int_0^{\frac{\pi}{2}} \sin^{2m}\theta\, d\theta$, (ii) $\int_0^{\frac{\pi}{2}} \cos^{2m}\theta\, d\theta$.

39. If m is a positive integer, find the value of

(i) $\int_0^{\frac{\pi}{2}} \sin^{2m+1}\theta\, d\theta$, (ii) $\int_0^{\frac{\pi}{2}} \cos^{2m+1}\theta\, d\theta$.

40. Evaluate $\int_0^{\infty} \frac{dx}{(1+x^2)^2}$ and $\int_0^{\frac{\pi}{4}} \tan^4 x\, dx$.

CHAPTER XVII.

DIFFERENTIATION AND INTEGRATION OF LOGARITHMS.

1. **To find the gradient of the tangent to the curve $y = \log_{10} x$, where it crosses the axis of x (i.e. at $x = 1$).**

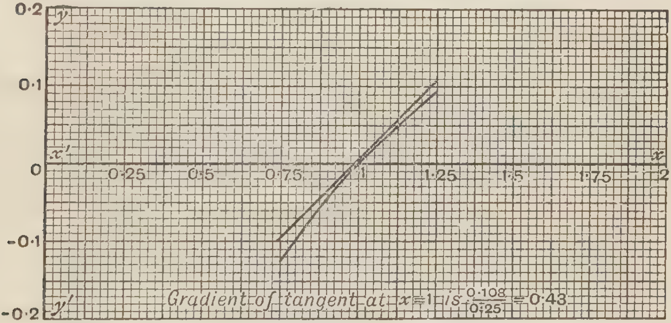

FIG. 172.

From the graph, we note that the gradient of the tangent at $x = 1$ is 0·43 very approximately.

We can verify this from a table of logarithms.

For $\log_{10} 1·00 = 0$

$\log_{10} 1·01 = 0·0043$

$$\therefore \frac{\log_{10} 1·01 - \log 1·00}{1·01 - 1·00} = \frac{0·0043}{0·01} = 0·43.$$

More approximate methods of calculation will show that the gradient of the tangent to the curve $y = \log_{10} x$ where it crosses the axis of x is 0·43429448, correct to *eight* places of decimals.

(See *Elementary Practical Mathematics*, by Prof. John Perry, p. 86.)

2. To find the ratio of the gradient at x to the gradient at X, for the curve $y = \log_{10} x$.

Let P and p be the points on the curve whose abscissae are X and x respectively.

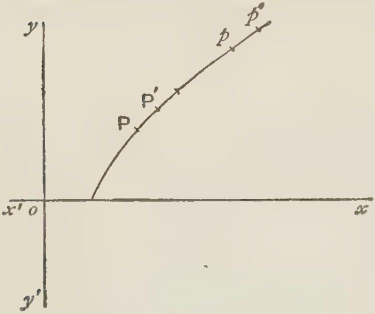

FIG. 173.

Let P′ and p' be points taken on the curve very near to P and p respectively, and such that

$$\frac{X'}{X} = \frac{x'}{x} \,\ldots\ldots\ldots(1)$$

where X′, x' are the abscissae of P′, p' respectively.

Hence

$$\frac{X' - X}{X} = \frac{x' - x}{x} \,\ldots\ldots(2)$$

Then we have:

$$\text{radient of chord } pp' = \frac{\log x' - \log x}{x' - x}$$

$$= \frac{\log \dfrac{x'}{x}}{x' - x}.$$

Also $$\text{gradient of chord PP}' = \frac{\log \dfrac{X'}{X}}{X' - X}.$$

Hence $$\frac{\text{gradient of chord } pp'}{\text{gradient of chord PP}'} = \frac{X' - X}{x' - x} \text{ by (1)}$$

$$= \frac{X}{x} \text{ by (2)}.$$

Hence, proceeding to the limit when the chords PP′, pp' become the tangents at P, p respectively, we obtain:

$$\frac{\text{gradient of tangent at } p}{\text{gradient of tangent at P}} = \frac{X}{x} \,\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

If we take P at the point where the curve crosses the axis of x, *i.e.* at $X = 1$, we know that the gradient of the tangent thereat is 0·4343 by Art. 1.

Hence, from (3),

$$\frac{\text{gradient of tangent at } p}{0{\cdot}4343}=\frac{1}{x}. \qquad \therefore \ \frac{d\log_{10}x}{dx}=\frac{0{\cdot}4343}{x}.$$

The numerical factor 0·4343 is very awkward in calculations, and hence we try to get rid of it as follows, by changing to a suitable logarithmic base.

Corollary. If we choose as our logarithmic base the number $e\equiv 2{\cdot}7182818285$ (correct to *ten* places of decimals), we find that the gradient of the tangent to the curve $y=\log_e x$ where it crosses the axis of x is unity. Hence, reasoning as above, we obtain:

$$\frac{d\log_e x}{dx}=\frac{1}{x}.$$

The above value for e may be conveniently obtained as follows:—Let us choose a new base e, at present unknown, so that $\frac{d}{dx}\log_e x=\frac{1}{x}$. Then by the elementary laws of logarithms $\log_{10} x=\log_{10} e\times\log_e x$.

$$\therefore \ \frac{d}{dx}\log_{10}x=\log_{10}e\times\frac{d}{dx}\log_e x. \qquad \therefore \ \frac{0{\cdot}4343}{x}=\log_{10}e\times\frac{1}{x}.$$

Hence $\log_{10}e=0{\cdot}4343. \quad \therefore \ e=2{\cdot}7183.$

Conversely: $\displaystyle\int\frac{dx}{x}=2{\cdot}3026\log_{10}x=\log_e x,$

where 2·3026 is the reciprocal of 0·4343, each correct to *four* places of decimals.

N.B. (1). *The integral of* x^{-1}.

The student will have noticed that hitherto we have given no formula for $\int x^{-1}\,dx$.

If we generalised from $\int x^n\,dx=\frac{x^{n+1}}{n+1}$, we should obtain $\int x^{-1}\,dx=\frac{x^{-1+1}}{-1+1}=\frac{x^0}{0}$, a nugatory result.

The result of Art. 2, however, enables us to solve the problem of evaluating $\int x^{-1}\,dx$.

N.B. (2). When the work of integration involves logarithms, it is best to be provided with tables having logarithms calculated to the base e, on account of the simplicity of the forms $\frac{d\log_e x}{dx}=\frac{1}{x}$ and $\int\frac{dx}{x}=\log_e x$, there being no numerical factors; thus much numerical computation is saved. We give some examples of using tables of logarithms to the bases 10 and e respectively.

We begin by giving examples of using tables of logarithms whose base is 10.

Example 1. *Evaluate* $\int_{1\cdot 3}^{5\cdot 2}\frac{1+x}{x}\,dx$.

We have
$$\int_{1\cdot 3}^{5\cdot 2}\frac{1+x}{x}\,dx=\int_{1\cdot 3}^{5\cdot 2}\left(\frac{1}{x}+1\right)dx$$
$$=\Big[2\cdot 3026\log_{10}x+x\Big]_{1\cdot 3}^{5\cdot 2}$$
$$=5\cdot 28.$$

Example 2. *Find the work done by a gas in expanding from* 3 *to* 4 *cub. ft., the pressure and volume being connected by the law* $pv=$ *constant, given that the pressure is* 2000 *lbs.-wt. per sq. ft. when the volume is* 0·3 *cub. ft.*

[*N.B.*—The work done is measured by $\int p\,dv$ ft.-lb.]

We have
$$pv=\text{constant}=2000\times 0\cdot 3$$
$$=600.$$
$$\therefore\ p=\frac{600}{v}.$$

Hence work done in ft.-lb.
$$=\int_3^4 p\,dv$$
$$=\int_3^4\frac{600}{v}\,dv$$

$$=600 \times 2{\cdot}3026 \log_{10} \tfrac{4}{3}$$
$$=600 \times 2{\cdot}3026 \times 0{\cdot}1249$$
$$=172{\cdot}6.$$

Work done by the gas in expanding is 172·6 ft.-lb.

Example 3. *A body is projected with a velocity of 2·5 ft. per sec., and moves in a medium where the retardation is proportional to the velocity. If the equation of motion is* $\frac{dv}{dt} = -3{\cdot}4v$, *find the formula giving v in terms of t. Find also the formula giving t in terms of v.*

We have $$\frac{dv}{dt} = -3{\cdot}4v.$$

$$\therefore\ -3{\cdot}4dt = \frac{dv}{v}.$$

$$\therefore\ -3{\cdot}4t + C = 2{\cdot}3026 \log_{10} v \text{.........................}(1)$$

But when $t=0$, $v=2{\cdot}5$.

$$\therefore\ C = 2{\cdot}3026 \log_{10} 2{\cdot}5 \text{.......................}(2)$$

From (1) and (2) by subtraction

$$-3{\cdot}4t = 2{\cdot}3026 \log_{10} \frac{v}{2{\cdot}5}.$$

$$\therefore\ t = 0{\cdot}677 \log_{10} \frac{5}{2v} = \frac{2}{3} \log_{10} \frac{5}{2v}.$$

Hence t is expressed in terms of v.

Again, $$\log_{10} \frac{5}{2v} = \frac{3t}{2}.$$

$$\therefore\ \frac{5}{2v} = 10^{\frac{3t}{2}}.$$

$$\therefore\ v = 2{\cdot}5 \times 10^{-1{\cdot}5t}.$$

And we have expressed v in terms of t.

We now give an example of using tables of logarithms whose base is e:—

Example 4. *Evaluate* $\int_{1\cdot7}^{9\cdot2} \frac{x}{1+x}\, dx$.

$$\int_{1\cdot7}^{9\cdot2} \frac{x}{1+x}\, dx = \int_{7\cdot1}^{9.2} \frac{1+x-1}{1+x}\, dx$$

$$=\int_{1\cdot 7}^{9\cdot 2}\left(1-\frac{1}{1+x}\right)dx$$

$$=\Big[x-\log_e(1+x)\Big]_{1\cdot 7}^{9\cdot 2}$$

$$=[9\cdot 2-\log_e 10\cdot 2-1\cdot 7+\log_e 2\cdot 7]$$

$$=7\cdot 5-\log_e 10\cdot 2+\log_e 2\cdot 7$$

$$=7\cdot 5-\{2\cdot 3026+0\cdot 0198\}+0\cdot 9933 \text{ (using tables to the base } e)$$

$$=6\cdot 1709.$$

3. The following integrals are important :—

I. $\int \tan x\,dx$.

We have $$\int \tan x\,dx=\int\frac{\sin x\,dx}{\cos x}$$
$$=-\int\frac{d\cos x}{\cos x}$$
$$=-\log_e\cos x.$$

II. $\int \cot x\,dx$.

We have $$\int \cot x\,dx=\int\frac{\cos x\,dx}{\sin x}$$
$$=\int\frac{d\sin x}{\sin x}$$
$$=\log_e\sin x.$$

III. $\int\frac{d\theta}{\sin\theta}$.

$$\int\frac{d\theta}{\sin\theta}=\int\frac{d\theta}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$=\int\frac{\sec^2\frac{\theta}{2}\,d\frac{\theta}{2}}{\tan\frac{\theta}{2}}\left(\text{dividing above and below by }\cos^2\frac{\theta}{2}\right)$$

$$= \int \frac{d \tan \frac{\theta}{2}}{\tan \frac{\theta}{2}}$$

$$= \log_e \tan \frac{\theta}{2}.$$

IV. $\int \frac{d\theta}{\cos \theta}$.

$$\int \frac{d\theta}{\cos \theta} = \int \frac{d\theta}{\sin\left(\frac{\pi}{2} + \theta\right)}$$

$$= \int \frac{d\left(\frac{\pi}{2} + \theta\right)}{\sin\left(\frac{\pi}{2} + \theta\right)}$$

$$= \log_e \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right).$$

4. Differentiation of powers of 10.

Consider $y = 10^x$.

$$\therefore \log_{10} y = x.$$

Hence, differentiating with respect to y, we get:

$$\frac{0{\cdot}4343}{y} = \frac{dx}{dy}.$$

$$\therefore \frac{dy}{dx} = 2{\cdot}3026 \times 10^x.$$

$$\textit{i.e.} \quad \frac{d\,10^x}{dx} = 2{\cdot}3026 \times 10^x.$$

More accurately, we have:

$$\frac{d\,10^x}{dx} = 2{\cdot}3025851 \times 10^x$$

and $$\int 10^x\,dx = 0{\cdot}43429448 \times 10^x.$$

Much more conveniently for purposes of computation, using tables of logarithms to the base e we have:

$$\frac{de^x}{dx} = e^x$$

and

$$\int e^x\,dx = e^x,$$

where $e \equiv 2{\cdot}7182818285$ (correct to 10 places of decimals).

Example 1. *Find the greatest value that the expression* $x\,e^{-x}$ *can have.*

On tracing the graph of the function $y = x\,e^{-x}$ we obtain—

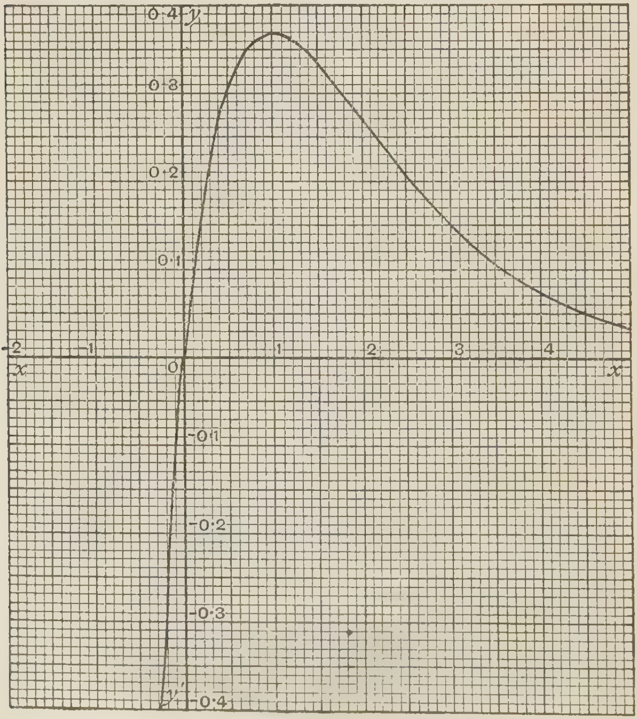

FIG. 174.

The maximum value of $y = xe^{-x}$ is got from the equation $\frac{dy}{dx} = 0$, *i.e.* $e^{-x} - xe^{-x} = 0$, *i.e.* $x = 1$, which is in agreement with the drawing.

Example 2. *Find the slope of the curve* $y = e^{-x} \sin x$ *at the point* $\frac{\pi}{2}$.

We have
$$\frac{dy}{dx} = e^{-x} \cos x - e^{-x} \sin x$$
$$= -e^{-\frac{\pi}{2}} \quad \text{when } x = \frac{\pi}{2}$$
$$= -0{\cdot}2078.$$
$$\therefore \tan \psi = +\frac{dy}{dx} = -0{\cdot}2078.$$
$$\therefore \psi = 168^\circ\, 16'.$$

Example 3. *Evaluate* $\int x e^x \, dx$.

We have
$$\int x e^x \, dx = \int x \, de^x$$
$$= x e^x - \int e^x \, dx$$
$$= x e^x - e^x$$
$$= e^x(x - 1).$$

EXAMPLES XVII.

1. If $y = \log_{10} x$, find $\frac{dy}{dx}$ when x has the values 0·5, 1·7, 3·48, 8·264, 100·1, 137·4, giving each answer correct to two places of decimals.

2. If $y = \log_{10} x$, calculate $\frac{d^2y}{dx^2}$ when x has the values 0·72, 2·31, 3·76, giving each answer correct to two places of decimals.

3. If $y = \log_{10} x$, prove that $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$.

4. Differentiate (i) $\log_{10}(1 + x^2)$,
(ii) $\log_{10}(1 + x + x^2)$,
(iii) $\log_{10}(3x + 4)$,

(iv) $\log_{10}(5x^2-3x-2)$,
(v) $\log_{10}\sin x$ (x in radians),
(vi) $\log_{10}\cos x$ (x in degrees),
(vii) $\log_{10}(3\sin x-4\cos x)$ (x in radians).

5. Evaluate $\frac{d}{dx}\log_{10}\sin x$, when $x=0{\cdot}7$ radians; $x=\frac{\pi}{3}$ radians; $x=\frac{\pi}{4}$ radians; $x=\frac{\pi}{6}$ radians; $x=1{\cdot}2$ radians, giving each answer correct to two places of decimals.

6. Differentiate $\log_{10}(\tan x)$ (x in radians).

7. Evaluate $\int_{0{\cdot}7}^{3{\cdot}9}\frac{dx}{x}$ and $\int_{1{\cdot}7}^{4{\cdot}6}\frac{dx}{x}$ (to two places of decimals)

8. Evaluate $\int_{0}^{5}\frac{dx}{1+x}$ ” ”

9. Evaluate $\int_{0}^{7{\cdot}3}\frac{dx}{2+x}$ ” ”

10. Evaluate $\int_{1{\cdot}3}^{7{\cdot}6}\frac{dx}{x-1}$ ” ”

11. Evaluate $\int_{0}^{6}\frac{dx}{2+3x}$ ” ”

12. Evaluate $\int_{0}^{7{\cdot}5}\frac{dx}{4+5x}$ ” ”

13. Evaluate $\int_{1{\cdot}2}^{6{\cdot}5}\frac{x\,dx}{x^2+1}$ ” ”

14. Evaluate $\int_{0{\cdot}5}^{8{\cdot}2}\frac{x\,dx}{2+3x^2}$ ” ”

15. Evaluate $\int_{0}^{1}\frac{x\,dx}{1+x}$ ” ”

16. Evaluate $\int_{2}^{10}\frac{dx}{x^2-1}$ ” ”

$$\left[\textit{Hint.}\quad \frac{1}{x^2-1}=\tfrac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right).\right]$$

17. Evaluate $\int_{4\cdot 1}^{7\cdot 6}\frac{dx}{x^2-5x+6}$ (to two places of decimals).

$$\left[\textit{Hint.}\quad \frac{1}{x^2-5x+6}=\frac{1}{(x-2)(x-3)}=\frac{1}{x-3}-\frac{1}{x-2}.\right]$$

18. Evaluate $\int_{2\cdot 3}^{8\cdot 1}\frac{dx}{(x+1)(x-2)}$.

19. Evaluate $\int_{1\cdot 3}^{4\cdot 6}\frac{x}{(x-1)(x+5)}dx$.

20. Trace the graph of $y=\frac{1-x}{1+x}$, and find the area intercepted in the positive quadrant between the axes and the curve.

21. Find the area intercepted in the positive quadrant between the axes and the curve $y=\frac{a-x}{a+x}$ where a is positive.

22. If $y=10^x$, find $\frac{dy}{dx}$ when $x=1$.

23. If $y=10^x$, find $\frac{dy}{dx}$ when $x=2{\cdot}3$.

24. If $y=10^{2x}$, find $\frac{dy}{dx}$ when $x=1$, $6{\cdot}2$.

25. If $y=10^{1\cdot 2x}$, find $\frac{dy}{dx}$ when $x=4{\cdot}6$.

26. Given that the equation $\frac{d^2y}{dx^2}=6y$ is satisfied by a solution of the form $y=10^{ax}$, find two possible values of a.

27. Given that the equation $1{\cdot}2\frac{d^2y}{dx^2}=7{\cdot}3y$ is satisfied by a solution of the form $y=10^{ax}$, find two possible values of a.

28. Express 37 as a power of 10 by means of your tables, and hence find $\frac{dy}{dx}$ when $y=37^x$ and $x=0{\cdot}3$.

29. Express 15·2 as a power of 10 by means of your tables, and hence find $\frac{dy}{dx}$ when $y=(15{\cdot}2)^x$ and $x=1{\cdot}8$.

30. If $y=\log_e x$, find $\frac{dy}{dx}$ when $x=2{\cdot}7$.

31. If $y=\log_e x$, find $\dfrac{dy}{dx}$ when $x=7{\cdot}2$.

32. If $y=x\log_e x$, find $\dfrac{dy}{dx}$ when $x=1{\cdot}7$.

33. If $y=x^2\log_e x$, find $\dfrac{dy}{dx}$ when $x=2{\cdot}1$.

34. If $y=(x^2+1)\log_e x$, find $\dfrac{dy}{dx}$ when $x=3{\cdot}4$.

35. If $y=e^x$, find $\dfrac{dy}{dx}$ when $x=0{\cdot}7$.

36. If $y=x\,e^x$, find $\dfrac{dy}{dx}$ when $x=1{\cdot}8$.

37. If $y=x^2e^x$, find $\dfrac{dy}{dx}$ when $x=2{\cdot}8$.

38. If $y=e^{3x}$, prove that y satisfies the differential equation

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}-3=0.$$

39. Given that the differential equation $\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}+6=0$ is satisfied by a solution of the form $y=e^{ax}$, find two values for a.

40. Evaluate $\displaystyle\int_0^{10} e^x\,dx$ to the nearest integer.

41. Evaluate $\displaystyle\int_0^{2} e^{3x}\,dx$ to two places of decimals.

42. Evaluate $\displaystyle\int_{1\cdot2}^{5\cdot7} e^x\,dx$ to two places of decimals.

43. Evaluate $\displaystyle\int_{0\cdot3}^{5\cdot1}(1+x)e^x\,dx$ to two places of decimals.

44. Calculate the minimum value of $x^2\log_{10}x$ to three places of decimals. (Q.P.M.S.T. Adapted.)

45. Prove that the area between the curve $y=a\,e^{-bx}\sin x$ and the axis of x, from $x=0$ to $x=\pi$, is $\dfrac{a}{b^2+1}(e^{-b\pi}+1)$. (Q.P.M.S.T.)

46. If in the last question $a=1$ and $b=0{\cdot}1$, prove that the decrease of area, when b is increased by a comparatively small quantity λ, is approximately $2{\cdot}6\lambda$. (Q.P.M.S.T.)

47. Prove that $\log_e x+\frac{1}{x}$ is not less than 1 for any positive value of x. (Q.P.M.S.T.)

48. Evaluate $\int_0^1 \frac{x^2}{4-x^2}\,dx$. (Q.P.M.S.T.)

49. Sketch the curves $y=e^x$ and $y=x\,e^x$ between $x=-1$ and $x=1$, and calculate the area enclosed between portions of these two curves and the axis of y. (Q.P.M.S.T.)

50. Evaluate the integral $\int e^{ax}\,x^2\,dx$. (Q.P.M.S.T.)

51. Evaluate the integral $\int_0^\infty \frac{r^4\,dx}{(r^2+x^2)^{\frac{5}{2}}}$. (Q.P.M.S.T.)

52. Show by sketches the general form of the curves

$$y=\frac{ax}{b+x}$$

$$y=a\,e^{-\frac{x}{b}}$$

$$y=a\sin\frac{x}{b}$$

for positive values of x. (Q.P.M.S.T.)

53. Evaluate $\int_0^1 \frac{x\,dx}{\sqrt{1-x^2}}$. (Q.P.M.S.T.)

54. Differentiate $e^{\sin^2 x}$ with respect to x. (Q.P.M.S.T.)

55. Find the fourth differential coefficient of $e^x \sin x$, with respect to x. (Q.P.M.S.T.)

56. Evaluate $\int x^2 \log_e x\,dx$. (Q.P.M.S.T.)

57. Evaluate $\int \frac{dx}{9-4x^2}$. (Q.P.M.S.T.)

58. A line rotates uniformly through a complete revolution about a fixed axis at one end, and its length increases uniformly from 1 in. to 2 in. during the revolution. Find the area swept out by the line. (Q.P.M.S.T.)

59. Draw the graph of $x \log_{10} x + \frac{2}{x}$ from $x = \frac{1}{2}$ to $x = 8$. Find the equation to determine the minimum value of this expression, and solve this equation graphically. (Q.P.M.S.T.)

60. Integrate $\frac{x}{\sqrt{x^2-1}}$ and $\frac{1}{\sqrt{x^2+x}}$, and find numerically the values of $\int_3^4 \frac{1}{x^2-4}\,dx$ and $\int_0^{\frac{\pi}{2}} x \sin x\,dx$. (Q.P.M.S.T.)

61. Find the differential coefficient with respect to x of (i) $\log \tan x$, (ii) $\frac{e^{x^2+1}}{x^2+1}$. (Q.P.M.S.T.)

62. Evaluate (i) $\int \frac{x+1}{(x+2)^2}\,dx$, (ii) $\int x \sin x\,dx$.

Prove that $\int_1^2 \frac{1}{x(x+1)}\,dx = 0{\cdot}288$. (Q.P.M.S.T.)

63. Differentiate with respect to x

$$\text{(i) } \log_e(\tan x + \sec x), \quad \text{(ii) } \frac{1+3x}{(1+4x+6x^2)^{\frac{1}{2}}}.$$

Find the slope of the curve $y = e^{-\frac{x}{10}} \sin 5x$ at the point $(\pi, 0)$. (Q.P.M.S.T.)

64. Evaluate (i) $\int \frac{1}{x^2+2x+5}\,dx$, (ii) $\int x\,e^x\,dx$. (Q.P.M.S.T.)

65. Differentiate $e^t \sin 3t$ with respect to t.

66. Differentiate $\frac{e^x}{\sin x}$ with respect to x.

67. The tangent at a point $P(x, y)$ on the curve $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ makes an angle ψ with the axis of x. Show that $y \cos\psi = a$.

If the tangent cuts the axis of y in the point U, show that $PU = \frac{xy}{a}$. (Q.P.M.S.T.)

68. Evaluate

(i) $\int_5^6 \frac{x^2\,dx}{x-2}$, (ii) $\int_0^{\frac{\pi}{3}} \sin x \cos 2x\,dx$, (iii) $\int x \log_e x\,dx$. (Q.P.M.S.T.)

69. Plot the curve $y^2x^2=36(x-2)(10-x)$, and show that the volume generated by revolving this curve about the axis of x is $119{\cdot}3\pi$. (Q.P.M S.T.)

70. Find the first and second differential coefficients of y with respect to x if

$$y=e^{\mathrm{K}x}(\mathrm{A}\sin\mu x+\mathrm{B}\cos\mu x),$$

and show that these satisfy the equation

$$\frac{d^2y}{dx^2}-2\mathrm{K}\frac{dy}{dx}+(\mathrm{K}^2+\mu^2)y=0.$$

Given that, when $x=0$, y has the value 4 and $\frac{dy}{dx}$ the value zero, find the values of A and B. (Q.P.M.S.T.)

71. Evaluate

$$\int_0^1\sqrt{1-x}\,dx,\quad \int_0^{\frac{\pi}{2}}\frac{dx}{3+5\cos x},\quad \int x^2e^{3x}\,dx.$$ (Q.P.M.S.T.)

72. Find the first and second differential coefficients of $x^2\log_e x$ with respect to x.

73. Evaluate the integrals

(i) $\int_0^1 x\log_e(1+x)\,dx$, (ii) $\int\frac{dx}{(x+a)(x+b)}\cdot$ (Q.P.M.S.T.)

74. Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $y=e^{ax}\sin bx$.

75. Evaluate $\int_0^1\frac{(x+1)\,dx}{(x+2)(x+3)}\cdot$ (Q.P.M.S.T.)

76. An electric condenser of capacity K farads and leakage resistance R ohms has been charged, and the voltage V is diminishing according to the law $\frac{dV}{dt}=-\frac{V}{KR}$.

Express V in terms of the time t sec. If $K=0{\cdot}8$ microfarad, if V is noted to be 30, and 15 sec. afterwards it is noted to be 26·43, find R.

77. $y=e^{ax}$ is a solution of $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$: find two values of a.

78. $y=e^{ax}$ is a solution of $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+8y=0$: find two values of a.

79. $y=x^n e^{ax}$ is a solution of $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0$: find values for n and a.

80. $y=x^n e^{ax}$ is a solution of $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=0$: find values for n and a.

81. Find $\int p\,dv$ if $pv^s=c$, a constant, (i) when $s=0{\cdot}8$, and (ii) when $s=1$. (Bd. of Educ.)

82. The rate (per unit increase of volume) of reception of heat by a gas is h, p is its pressure, and v its volume ; γ is a known constant.

(i) If $pv^s=c$, s and c being constants, find h if $h=\frac{1}{\gamma-1}\left\{v\frac{dp}{dv}+\gamma p\right\}$.

(ii) If h is always 0, what must s be ? (Bd. of Educ.)

83. A particle moves in a straight line under the influence of a retarding force which acts so that the law of motion is $\frac{dv}{dt}=-2{\cdot}3v$, where v is measured in cm. per sec. and t in sec. The particle starts from the origin with a velocity of 10 cm. per sec.

(i) Find the formula connecting v and t.

(ii) Find at what time the particle is moving with half its original velocity.

(iii) Find the velocity of the particle after 2 sec.

(iv) Find the formula connecting v and s.

(v) Find how far the particle is from the origin when its velocity is 7 cm. per sec.

(vi) Find the formula connecting s and t.

(vii) Find how far the particle is from the origin after 3 sec.

84. A particle moves in a straight line under the influence of a retarding force so that the law of motion is

$$\frac{dv}{dt}=-\mathrm{K}v,$$

where space and time are measured in cm. and sec. respectively.

(i) If the velocity of the particle is 12 cm. per sec. at the start of the motion, and 8 cm. per sec. after 2 sec., calculate K.

(ii) Find how far the particle is from the origin after 3 sec., and what velocity it has then.

(iii) Find at what time the particle is 1 cm. from the origin.

85. If p dynes per sq. cm. be the pressure of the atmosphere at a height of x cm., then it is known that

$$\frac{dp}{dx}=-\frac{p}{\text{H}}, \qquad \text{where H}=7{\cdot}988\times10^5\times(1+0{\cdot}00366t)$$

at a temperature $t°$ Centigrade in the latitude of Greenwich. Find the height at which the atmospheric pressure is halved, the temperature remaining constant at 5° C.

86. In question 85 the pressure at the ground level is found to be 10^6 dynes per sq. cm. at a temperature of 10° C. What is the pressure at a height of 1 kilometre, supposing the temperature unchanged?

87. Newton's Law of Cooling states: "The rate of cooling of a body is proportional to the difference between the temperature of the body and that of its surroundings," *i.e.* if the temperature of a body is T° Centigrade at t sec. from a given instant, and if S° Centigrade be the temperature of its surroundings, then $\frac{d\text{T}}{dt}=-k(\text{T}-\text{S})$, where k is a constant.

If a body takes 5 minutes to cool from 80° C. to 70° C., how long will it take to cool from 30° C. to 20° C., the temperature of the surroundings being 11° C.?

88. A body cools from 20·3° C. to 19·8° C. in 182 sec.; it cools from T° C. to 24·1° C. in 101 sec. Assuming Newton's Law of Cooling, find T, the temperature of the surroundings in both cases being 14·7° C.

89. A metal cylinder long in comparison with its radius has a circular hole right through its axis. Through this hole passes a current of superheated steam, while the outside of the cylinder is cooled by cold water circulating spirally round it. If the cylinder is l cm. in length and k is the conductivity of the metal of which it consists, while Q calories of heat is supplied per second by the steam, then if T° Centigrade is the temperature of the metal at a radial distance r cm. from the axis it can be proved that $-k\frac{d\text{T}}{dr}=\frac{\text{Q}}{2\pi rl}$. Find T in terms of r, assuming that the radius of the circular hole is 1 cm. and the temperature of the steam is $\theta°$ Centigrade.

90. In question 89, if $k=1{\cdot}05$, $l=30$, $\text{Q}=1{\cdot}94\times10^4$, $\theta=120$, find the temperature of the metal 3 cm. from the axis.

91. In question 89, if the bar be of iron whose $k=0{\cdot}164$, $l=28$ cm., $\text{Q}=3\times10^3$, $\theta=150°$, find the distance from the axis at which the temperature is 40° C.

92. Find the area intercepted between the curve $xy=49$ and the ordinates $x=7$ and $x=12$ and the axis of x.

93. 1 gram of hydrogen occupies a volume of 11,610 c.c. at 12° C., and exerts a pressure of 1·014 megadyne to the sq. cm. Find in joules the work done in compressing the gas to half its volume without rise in temperature, the gas obeying Boyle's Law.

94. In a submarine cable the radius of the cross-section of the conducting wire is r_1 cm., and the radius of the cross-section of the insulating material is r_2 cm. If V is the electrical potential at a distance r cm. from the axis of the cable ($r_2 > r > r_1$), and Q the charge on 1 cm. unit length of cable, while K is the specific inductive capacity of the insulating material, then it is known that $\frac{dV}{dr} = -\frac{2Q}{Kr}$, the potential of the outside of the cable being zero.

Find the potential of the conducting wire of the cable. Defining the capacity per unit length of the cable as $\frac{Q}{V}$, find the capacity per unit length of the above cable.

95. A cable consists of a wire 2·2 mm. in diameter and insulated with guttapercha to a thickness of 3·5 mm. For guttapercha K = 4·25. What is the capacity of 1 kilometre of the cable?

96. If μ be the coefficient of friction between a rope and a circular drum upon which it is wound, and if T lb.-wt. be the tension of the rope at a point such that the length of rope between it and the free end at which the tension is least subtends θ radians at the axis of the drum, then we know that $\frac{dT}{d\theta} = \mu T$.

Express T in terms of μ and θ and T_0, the least tension of the rope at a free end.

97. A weight of 50 lb. is being lowered slowly down a shaft by means of a rope which passes over a fixed circular drum and is then held horizontally by a man. If the coefficient of friction between the rope and the drum be 0·227, find the pull the man has to support.

98. A rope makes $3\frac{1}{2}$ complete revolutions on a fixed wooden drum, and one end is held by a man who can exert a pull of half a hundredweight. If the coefficient of friction between the rope and the drum be 0·208, what pull at the other end of the rope can the man hold?

99. What is the coefficient of friction between a rope and a drum when a man pulling 40 lb.-wt. can hold a team of horses pulling 1 ton, the rope being wound 3 times round the drum?

100. How many turns on an iron bollard must a man pulling 56 lb.-wt. take in order to hold a boat making a pull of 5 tons? The coefficient of friction of rope on iron is 0·179.

CHAPTER XVIII.

EXPANSIONS AND CALCULATIONS.

1. Consider the trigonometrical functions. It is easy to calculate their values corresponding to 0°, 30°, 45°, 60°, 90° by the methods of elementary geometry. It is not easy in general to calculate intermediate values. The object of this chapter is to explain how to calculate intermediate values by the method of Brook Taylor in his book *Methodus Incrementorum Directa et Inversa* (published in London, 1715–1717), and Colin Maclaurin, Professor of Mathematics at Edinburgh, in his treatise on Fluxions (published in 1742). The method can be used to deal with other functions as well as those presenting themselves in trigonometry.

Use of the tangent to a curve for approximative purposes.

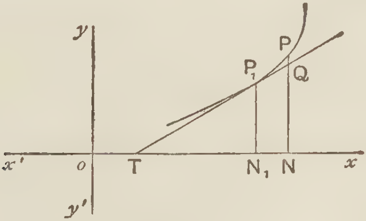

FIG. 175.

Let us consider the tangent at the point P_1 on the above curve.

For a considerable distance along the curve, the tangent and the curve appear to coincide. Hence we may take the ordinates of points on the tangent to have the same values as ordinates of points on the curve for approximative purposes. On the other hand, if we move away from P_1 some considerable distance, the tangent and the curve no longer appear to coincide. In fact, the error committed in taking QN as an approximate value for PN is plainly PQ, which soon becomes appreciable in value as PN moves away from P_1N_1.

We therefore form the equation of the tangent to the curve at the point P_1, and use this as very approximately representing the curve in the neighbourhood of P_1.

We shall now give some examples of taking the tangent to represent the curve at a particular point, and so finding approximate values.

Example 1. *It being known that* $\sin 30° = 0{\cdot}5$, *required to calculate* $\sin 30°\ 27'$.

We have $$y = \sin x \text{ (x in radians).}$$

$$\therefore \frac{dy}{dx} = \cos x.$$

Hence the value of $\frac{dy}{dx}$ when $x\,(=30°) = \frac{\pi}{6}$ radians, is $\frac{\sqrt{3}}{2}$.

The point on the curve corresponding to 30° is $\left(\frac{\pi}{6}, 0{\cdot}5\right)$, radian measure being used.

Hence the equation of the geometrical tangent at the point $\left(\frac{\pi}{6}, 0{\cdot}5\right)$ is

$$(y - 0{\cdot}5) = \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) \dots\dots\dots\dots (1)$$

Thus when $x = 30°\ 27' = \frac{\pi}{6} + 0{\cdot}007854$ radians,

$$y = 0{\cdot}5 + 0{\cdot}866 \times 0{\cdot}007854$$
$$= 0{\cdot}5068015.$$

N.B.—The Tables give 0·5067863.

Example 2. *It being known that* $\log_{10} \tan 45° = 0$, *required to calculate by the method of the "approximating-tangent"* $\log_{10} \tan 45°\ 49'$, $\log_{10} \tan 46°$, $\log_{10} \tan 50°$, $\log_{10} \tan 55°$, $\log_{10} \tan 60°$, *and to tabulate the results thus obtained.*

Consider $y = \log_{10} \tan x$ (x in radians).

Plainly $y=0$ when $x = 45° = \frac{\pi}{4}$ radians.

$$\therefore \left(\frac{\pi}{4}, 0\right) \text{ is a point on the curve } \dots\dots\dots\dots\dots\dots (1)$$

Also
$$\frac{dy}{dx} = 0{\cdot}4343 \sec x \operatorname{cosec} x$$
$$= 0{\cdot}8686 \text{ when } x = \frac{\pi}{4} \dots\dots\dots\dots\dots\dots\dots\dots (2)$$

Hence by (1) and (2) the equation to the geometrical tangent at the point $\left(\frac{\pi}{4}, 0\right)$ is

$$y = 0{\cdot}8686\left(x - \frac{\pi}{4}\right) \dots\dots\dots\dots\dots\dots\dots\dots (3)$$

If we therefore insert instead of $x - \frac{\pi}{4}$ the radian values for 49′, 1°, 5°, 10°, 15°, viz. $\frac{\pi}{180} \times \frac{49}{60}$, $\frac{\pi}{180}$, $\frac{\pi}{36}$, $\frac{\pi}{18}$, $\frac{\pi}{12}$ radians respectively, in (3), and make the necessary calculations, we get the following table :—

x	45° 49′	46°	50°	55°	60°
Calculated value of $\log_{10} \tan x$	0·0124	0·0152	0·0758	0·1516	0·2274
Value of $\log_{10} \tan x$ from the Tables	0·0124	0·0152	0·0762	0·1548	0·2386
Number of places of decimals to which you are correct	4	4	3	2	1 One "out" in second place.

We thus see that as we get further from the point $\left(\frac{\pi}{4}, 0\right)$ along the curve, the computation by the method of the approximating-tangent becomes less and less accurate.

Second approximating-curves.

If we wish a closer approximation at the point (x_1, y_1) on the curve, than that given by the tangent $(y-y_1)=m(x-x_1)$, we use a "second approximating-curve" of the form

$$(y-y_1)=m(x-x_1)+p(x-x_1)^2 \quad\text{.................}(1)$$

where m and p are constants. To find the numerical values of m and p, we calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and then find the values of these for the function at the point (x_1, y_1). The values of m and p are thus easily obtained, since the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are the same for both the original curve and the second approximating-curve at the point (x_1, y_1).

By (1), $$\frac{dy}{dx}=m+2p(x-x_1) \quad\text{.......................}(2)$$

By (2), $$\frac{d^2y}{dx^2}=2p \quad\text{....................................}(3)$$

From (2), $\left(\frac{dy}{dx}\right)$ has the value m at (x_1, y_1), as is easily seen by putting $x=x_1$ in (2).

From (3), $2p$ has the value $\frac{d^2y}{dx^2}$ at the point (x_1, y_1).

Thus the numerical values of m and p are easily found when the function to be evaluated is given.

Example 1. *Given that* $\cos 30°=\frac{\sqrt{3}}{2}$, *required to calculate* $\cos 30° 27'$, *using the second approximating-curve.*

Since $30°=\frac{\pi}{6}$, the point on the curve $y=\cos x$ (x in radians) under consideration is $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$.

Hence and
$$\left.\begin{aligned}\frac{dy}{dx}&=-\frac{1}{2}\\ \frac{d^2y}{dx^2}&=-\frac{\sqrt{3}}{2}\end{aligned}\right\}\text{ when } x=\frac{\pi}{6}.\ldots\ldots\ldots\ldots(1)$$

Let the equation to the second approximating-curve at the above point be:

$$\left(y-\frac{\sqrt{3}}{2}\right)=m\left(x-\frac{\pi}{6}\right)+p\left(x-\frac{\pi}{6}\right)^2.$$

Hence, differentiating the equation of the second approximating-curve, we get:

$$\frac{dy}{dx}=m+2p\left(x-\frac{\pi}{6}\right)$$

and
$$\frac{d^2y}{dx^2}=2p.$$

Hence, for the second approximating-curve:

$$\left.\begin{aligned}\frac{dy}{dx}&=m\\ \frac{d^2y}{dx^2}&=2p\end{aligned}\right\}\text{ when } x=\frac{\pi}{6}\ldots\ldots\ldots\ldots(2)$$

Hence, identifying (1) and (2), since $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ are each to have the same values for the given curve and also for the second approximating-curve, we get:

$$m=-\frac{1}{2}$$

$$p=-\frac{\sqrt{3}}{4}$$

Thus, the required equation to the second approximating-curve is:

$$\left(y-\frac{\sqrt{3}}{2}\right)=-\frac{1}{2}\left(x-\frac{\pi}{6}\right)-\frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2\ldots\ldots\ldots\ldots(3)$$

Now the interval under consideration is 27′, *i.e.* $\frac{27}{60}\times\frac{\pi}{180}$ radians, *i.e.* $\frac{\pi}{400}$ radians.

Hence, putting $x-\frac{\pi}{6}=\frac{\pi}{400}$ in (3), we obtain:

$$y-\frac{\sqrt{3}}{2}=-\frac{1}{2}\left(\frac{\pi}{400}\right)-\frac{\sqrt{3}}{4}\left(\frac{\pi}{400}\right)^2,$$

$$\textit{i.e.}\ y=0{\cdot}8620717.$$

From the Tables we obtain :

$$\cos 30^\circ\ 27'=0{\cdot}8620717.$$

Example 2. *Given that* tan 45° = 1, *calculate by the above method* tan 50° *and* tan 55°.

$$\because\ 45^\circ=\frac{\pi}{4}$$

and $$\tan 45^\circ=1,$$

$\therefore$ the point $\left(\frac{\pi}{4}, 1\right)$ lies on the curve $y=\tan x$, where x is measured in radians.

Let us take as our second approximating-curve :

$$y-1=m\left(x-\frac{\pi}{4}\right)+p\left(x-\frac{\pi}{4}\right)^2 \ \ldots\ldots\ldots\ldots(1)$$

As before, $$\frac{dy}{dx}=m+2p\left(x-\frac{\pi}{4}\right) \ \ldots\ldots\ldots\ldots(2)$$

and $$\frac{d^2y}{dx^2}=2p. \ \ldots\ldots\ldots\ldots(3)$$

$$\therefore\ m=\text{the value of }\frac{dy}{dx}\text{ when }x=\frac{\pi}{4}\ldots\ldots\ldots\ldots(4)$$

and $$2p=\text{the value of }\frac{d^2y}{dx^2}\text{ when }x=\frac{\pi}{4}\ldots\ldots\ldots\ldots(5)$$

But $$y=\tan x.$$

$$\therefore\ \frac{dy}{dx}=\sec^2 x$$

$$=2\text{ when }x=\frac{\pi}{4}\ \ldots\ldots\ldots\ldots(6)$$

Also $$\frac{d^2y}{dx^2}=2\sec^2 x\tan x$$

$$=4\text{ when }x=\frac{\pi}{4}\ \ldots\ldots\ldots\ldots(7)$$

Hence, by (1), (4), (5), (6), (7), the equation to our "approximating-curve" becomes :

$$y-1=2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2 \ \ldots\ldots\ldots\ldots(8)$$

CASE I. tan 50°.

In this case the interval is 5°, *i.e.* $\frac{\pi}{36}$ radians. Hence we put $x-\frac{\pi}{4}=\frac{\pi}{36}$ in (8), obtaining:

$$y-1=2\cdot\frac{\pi}{36}+2\left(\frac{\pi}{36}\right)^2.$$

$$\therefore\ y=1\cdot1897638.$$

From the Tables, tan 50° = 1·1917536. The calculation is therefore correct only to *two* places, and is only *two* out in the third place.

CASE II. tan 55°.

In this case the interval is 10°, *i.e.* $\frac{\pi}{18}$ radians. Hence we put $x-\frac{\pi}{4}=\frac{\pi}{18}$, and obtain:

$$y-1=2\cdot\frac{\pi}{18}+2\left(\frac{\pi}{18}\right)^2$$

$$=0\cdot3490659+0\cdot0609240.$$

$$\therefore\ y=1\cdot4099899.$$

From the Tables we get tan 55° = 1·4281480, a result correct only to *one* place of decimals.

Note I.—For a third approximating-curve at the point (x_1, y_1), we use a curve of the form:

$$(y-y_1)=m(x-x_1)+p(x-x_1)^2+q(x-x_1)^3$$

and proceed as before, making the values for $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ at (x_1, y_1) coincide with those of the given curve.

Note II.—If the tangent at (x_1, y_1) be parallel to the axis of x, m has the value zero. The second and third, etc., approximating-curves then lack the terms in $(x-x_1)$, and have the forms:

$$y=p(x-x_1)^2+q(x-x_1)^3, \text{ etc.}$$

Note III.—If the tangent at (x_1, y_1) is parallel to the axis of y, m has the value ∞, and hence we cannot assume the expansion in $(x-x_1)$, $(x-x_1)^2$, $(x-x_1)^3$, etc., for the approximating-curves. For approximating purposes, in the case when the tangent at (x_1, y_1) is parallel to the axis of y, we take our approximating-curves in the form $(x-x_1)=m'(y-y_1)+p'(y-y_1)^2+q'(y-y_1)^3$, etc., where $m'=0$.

$y = \log x.$

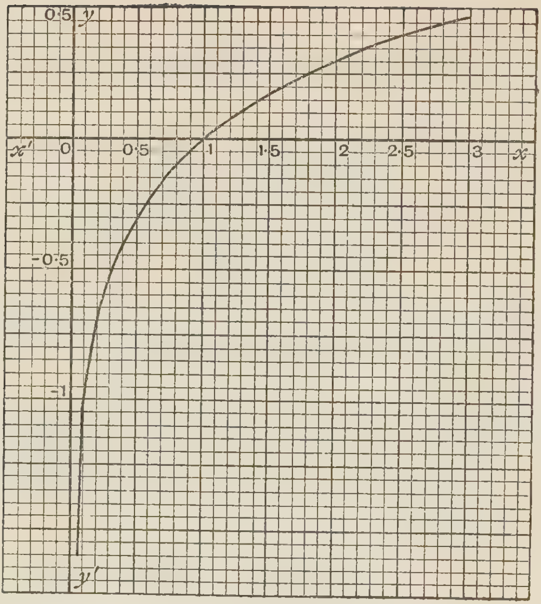

FIG. 176.

$y = \frac{2x+3}{x-1}.$

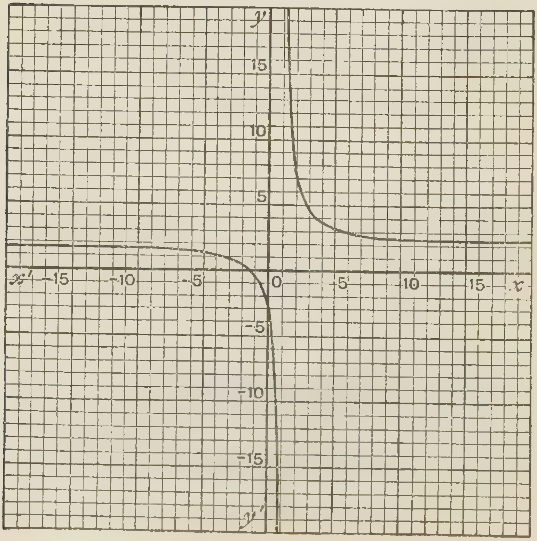

FIG. 177.

Note IV.—Where the value of y is infinite, as in the case of the curve $y=\log x$ when $x=0$ (fig. 176), and of $y=\frac{2x+3}{x-1}$ when $x=1$ (fig. 177), we cannot draw geometrical tangents to the curves at the points corresponding to $x=0$ and $x=1$ respectively, and hence the above methods of approximation at such failing cases cannot be carried out. The methods for such cases are too hard for discussion here.

2. The graphs of approximating-curves.

We shall now exhibit some diagrams showing the rapidity with which approximating-curves approach the curve itself, and also giving some idea of the range of values of x for which the approximations hold good.

Example 1. *Find the first, second, and third approximating-curves to $y=\cos x°$ at the point $(0, 1)$, and trace these curves all on the same graph from $x=0°$ to $x=90°$. Make any remarks that the graphs call forth regarding the goodness of the various approximations, and throughout what intervals the graphs show them to be fairly good approximations.*

The first approximating-curve is $y=1$, and the second and third approximating-curves are the same, viz. $y=1-\frac{1}{2}\left(\frac{\pi}{180}\right)^2 x^2$. We therefore obtain the subjoined diagram :—

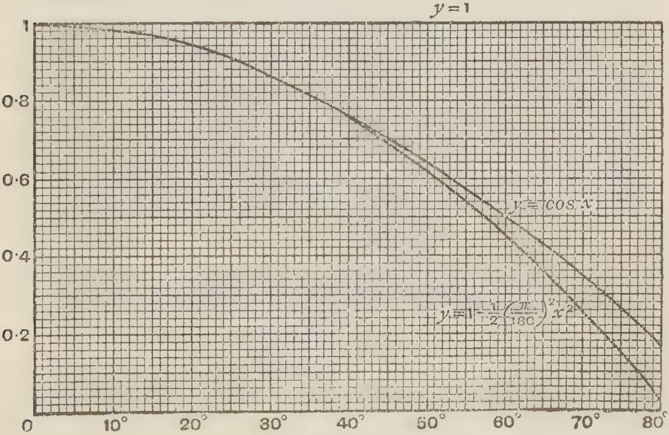

FIG. 178.

From the graph we see that $y=1$ is a *very fair* approximation for $y=\cos x$ from 0° to 10°, and then a considerable divergence begins; but $y=1-\frac{1}{2}\left(\frac{\pi}{180}\right)^2 x^2$ is a *good* approximation for $y=\cos x$ from 0° to 40°, and a *very fair* one from 40° to 50°.

Example 2. *Trace the graph of $y=\log_e x$ from $x=1$ to $x=9$ (on as big a scale as your squared paper will allow). Draw the first, second, and third approximating-curves at the point (1, 0) where the curve cuts the axis of x. Remark on the rapidity of the approximation of the successive approximating-curves.*

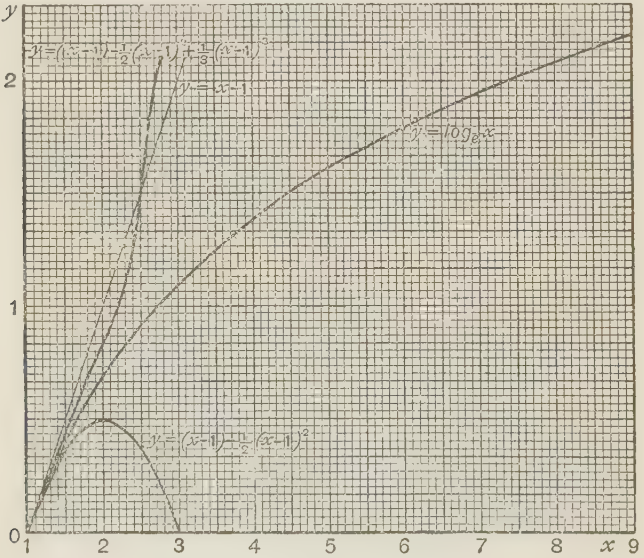

FIG. 179.

The first, second, and third approximating-curves at the point (1, 0) are respectively:

$$y=x-1,\quad y=(x-1)-\tfrac{1}{2}(x-1)^2$$

and

$$y=(x-1)-\tfrac{1}{2}(x-1)^2+\tfrac{1}{3}(x-1)^3.$$

The graphs show that all are *good* approximations between the values 1 and 1·5 of x, and *fairly good* between 1·5 and 2, but from $x=2$ upwards are worthless.

3. To find the first, second, third, etc., approximating-curves to the curve y = f(x) at the point (x$_1$, y$_1$) on the curve.

Let us denote $\frac{d f(x)}{dx}$, $\frac{d^2 f(x)}{dx^2}$, $\frac{d^3 f(x)}{dx^3}$, etc., by $f'(x)$, $f''(x)$, $f'''(x)$, etc.

We shall find the third approximating-curve by way of illustration, but the method is perfectly general.

Let the third approximating-curve be:

$$y - y_1 = m(x-x_1) + p(x-x_1)^2 + q(x-x_1)^3.$$

Hence
$$\frac{dy}{dx} = m + 2p(x-x_1) + 3q(x-x_1)^2$$
$$\frac{d^2y}{dx^2} = \quad 2p \quad + 6q(x-x_1)$$
$$\frac{d^3y}{dx^3} = \quad 6q$$

i.e. $\frac{dy}{dx}$ has the value m when $x=x_1$ ....(1)

$\frac{d^2y}{dx^2}$ „ „ $2p$ „ „ ....(2)

$\frac{d^3y}{dx^3}$ „ „ $6q$ „ „ ....(3)

But the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ are going to be the same at the point (x_1, y_1) for the third approximating-curve and the given curve $y=f(x)$.

Hence $m = f'(x_1)$ by..........................(1)

$p = \frac{1}{2} f''(x_1)$ by ....(2)

$q = \frac{1}{6} f'''(x_1)$ by ....(3)

Hence the third approximating-curve to $y=f(x)$ at the point (x_1, y_1) is:

$$y - y_1 = (x-x_1) f'(x_1) + \tfrac{1}{2}(x-x_1)^2 f''(x_1) + \tfrac{1}{6}(x-x_1)^3 f'''(x_1)$$

or, as it may be written, since $y_1 = f(x_1)$:

$$y = f(x_1) + (x-x_1) f'(x_1) + \tfrac{1}{2}(x-x_1)^2 f''(x_1) + \tfrac{1}{6}(x-x_1)^3 f'''(x_1).$$

Corollary. If $\underline{|n}$ denote the product of the first n natural numbers, viz. 1, 2, 3, 4, ... n, the n^{th} approximating-curve to $y=f(x)$ at the point (x_1, y_1) is:

$$y=f(x_1)+(x-x_1)f'(x_1)+\frac{(x-x_1)^2}{\underline{|2}}f''(x_1)+\frac{(x-x_1)^3}{\underline{|3}}f'''(x_1)$$
$$+\ldots+\frac{(x-x_1)^n}{\underline{|n}}f^n(x_1).$$

4. To estimate the error in using any given approximating-curve.

We shall require the following results:—

I. *If* A *and* B *be two points on a curve, then at some point on the curve between* A *and* B *the tangent to the curve is parallel to chord* AB.

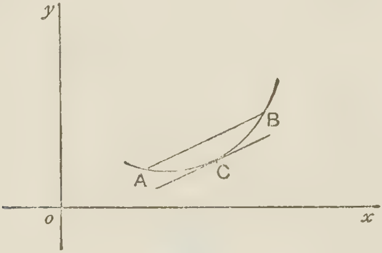

FIG. 180.

This result is evident from the above figure, since the tangent at C is plainly parallel to AB.

Let $y=f(x)$ be the equation to the above curve, and let the abscissae of A, B, C be a, b, c respectively. Then the gradient of the tangent at C is $f'(c)$, and the gradient of the line AB is $\frac{f(b)-f(a)}{b-a}$.

Hence $$\frac{f(b)-f(a)}{b-a}=f'(c),$$

where c lies between a and b.

II. *The n^{th} moment of an area about a given line.*

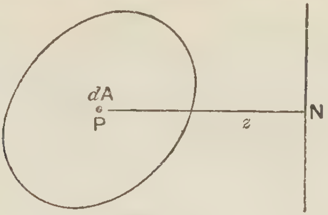

FIG. 181.

If dA be a small portion of a given area and z the perpendicular distance of dA from the given line, we define $\int z^n dA$ (taken all over the given area) as the n^{th} moment of the given area about the assigned line. The n^{th} moment of the area is the sum of all the small portions of the given area, each multiplied by the n^{th} power of its perpendicular distance from the given line.

To find the n^{th} moment of the area of a trapezium about the longer of its parallel sides.

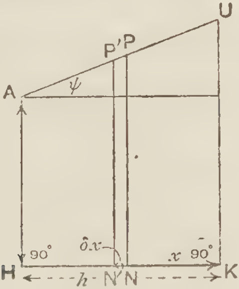

FIG. 182.

Let AHKU be the given trapezium, right-angled at H and K, and let PN, P′N′ be lines parallel to AH and very close to one another. Let $\text{KN} = x$ and $\text{NN}' = \delta x$. Let $\text{HK} = h$, $\text{AH} = p$, and the gradient of $\text{AU} = \tan\psi$. Then the n^{th} moment of the area PNN′P′ about KU is plainly $\text{PN}\delta x \times \text{NK}^n$ very approximately.

i.e. $(p + h\,\overline{-x}\tan\psi)\,\delta x \times x^n$ very approximately.

Hence the n^{th} moment of the area of the trapezium AHKU about KU is:

$$\int_0^h x^n(p+\overline{h-x}\tan\psi)\,dx=\frac{ph^{n+1}}{n+1}+\frac{h^{n+2}\tan\psi}{(n+1)(n+2)}.$$

III. *To find the exact value of the error in using the n^{th} approximating-curve at the point $x=a$ to calculate the value of $f(x)$ at $x=b$.*

We have

$$\begin{aligned}
f(b)-f(a)&=\int_a^b f'(x)\,dx\\
&=-\int_a^b f'(x)\,d(b-x)\\
&=-\Big[f'(x)\,(b-x)\Big]_a^b+\int_a^b(b-x)f''(x)\,dx \text{ (integrating by parts)}\\
&=(b-a)f'(a)-\tfrac{1}{2}\int_a^b f''(x)\,d(b-x)^2\\
&=(b-a)f'(a)-\tfrac{1}{2}\Big[f''(x)\,(b-x)^2\Big]_a^b+\tfrac{1}{2}\int_a^b(b-x)^2f'''(x)\,dx\\
&=(b-a)f'(a)+\tfrac{1}{2}(b-a)^2f''(a)+\tfrac{1}{2}\int_a^b(b-x)^2f'''(x)\,dx\\
&=(b-a)f'(a)+\tfrac{1}{2}(b-a)^2f''(a)-\frac{1}{2.3}\int_a^b f'''(x)\,d(b-x)^3\\
&=(b-a)f'(a)+\tfrac{1}{2}(b-a)^2f''(a)+\frac{1}{2.3}(b-a)^3f'''(a)\\
&\qquad+\frac{1}{2.3}\int_a^b(b-x)^3f''''(x)\,dx.
\end{aligned}$$

Proceeding in this way, we get the following identity:

$$\begin{aligned}
f(b)-f(a)\equiv(b-a)f'(a)&+\frac{(b-a)^2}{\underline{|2}}f''(a)+\frac{(b-a)^3}{\underline{|3}}f'''(a)+\dots\\
&+\frac{(b-a)^nf^{(n)}(a)}{\underline{|n}}+\frac{1}{\underline{|n}}\int_a^b(b-x)^nf^{(n+1)}(x)\,dx\,\ldots\ldots\ldots(1)
\end{aligned}$$

Hence the exact value of the error committed in using the n^{th} approximating-curve at $x=a$ to calculate the value of $f(x)$ at

$x=b$ is $\dfrac{1}{\lfloor n}\displaystyle\int_a^b (b-x)^n f^{(n+1)}(x)\,dx$, i.e. *the n^{th} moment about the line $x=b$ of the area included between the ordinates $x=a$ and $x=b$, the curve $y=f^{(n+1)}(x)$, and the axis of x, divided by $\lfloor n$.*

In most cases it is impossible to calculate this error *exactly*, and in such cases an upper limit to the error has to be found for practical purposes, by either of the following two methods, whichever is the more convenient for the given case.

I. Lagrange's Method.

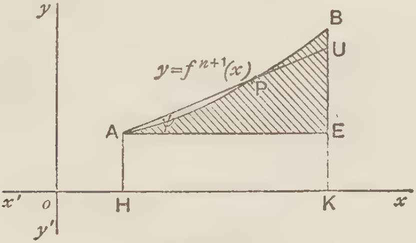

FIG. 183.

Let, in the above diagram, AB represent the arc of $y=f^{(n+1)}(x)$ lying between $x=a$ and $x=b$, and let AE be parallel to HK. Then it is plain that we can replace the n^{th} moment of the shaded area AEB, lying between AE, EB and the curve, by the n^{th} moment of the triangle AEU where AU cuts the curve at P somewhere between A and B, and where the n^{th} moment of the shaded part PUB cut off is equal to the n^{th} moment of the unshaded portion lying between the straight line AP and the arc AP. But, by II., the n^{th} moment of the trapezium AHKU about KU is:

$$\frac{\text{AH}\,.\,\text{HK}^{n+1}}{n+1}+\frac{\text{HK}^{n+2}\tan\psi}{(n+1)(n+2)}.$$

$$\therefore\ \int_a^b (b-x)^n f^{(n+1)}(x)\,dx \equiv \frac{f^{(n+1)}(a)(b-a)^{n+1}}{n+1}+\frac{(b-a)^{n+2}\tan\psi}{(n+1)(n+2)},$$

$$i.e.\ \frac{(b-a)^{n+1}f^{(n+1)}(a)}{n+1}+\frac{(b-a)^{n+2}f^{(n+2)}(c)}{(n+1)(n+2)}\ \ldots\ldots\ldots\ldots(2)$$

since at some point on the curve $y = f^{(n+1)}(x)$ lying between A and P the tangent to the curve is parallel to the chord AP. Let the abscissa of this point be c,

$$\therefore \tan\psi = \frac{df^{(n+1)}(x)}{dx} \text{ when } x = c.$$

Hence, by (1) and (2),

$$f(b) - f(a) \equiv (b-a)f'(a) + \frac{(b-a)^2}{\underline{|2}}f''(a) + \frac{(b-a)^3}{\underline{|3}}f'''(a) + \dots$$
$$+ \frac{(b-a)^{n+1}f^{(n+1)}(a)}{\underline{|n+1}} + \frac{(b-a)^{n+2}f^{(n+2)}(c)}{\underline{|n+2}}.$$

N.B.—Usually $f^{(n+2)}(c)$ cannot be calculated exactly because c is not known, except that it lies between a and b. What is usually done is therefore to obtain *by inspection* the biggest value for $f^{(n+2)}(x)$ within the interval $x = a$ and $x = b$, and to use this in obtaining an upper limit to the error required.

We therefore obtain the following practical rule:—

If the n^{th} approximating-curve at the point A *on the curve $y = f(x)$ be used in order to calculate the value of the function at* B, *then the error committed is certainly less than* $\frac{G(b-a)^{n+1}}{\underline{|n+1}}$, *where* G *is any convenient number (obtained by inspection or otherwise) greater than every value of* $\frac{d^{n+1}y}{dx^{n+1}}$ *within the interval* $a - b$.

II. CAUCHY'S METHOD.

Let us trace the curve $y = (b-x)^n f^{(n+1)}(x)$ from $x = a$ to $x = b$. Then plainly a line can be drawn parallel to the axis of x and cutting this curve at a point whose abscissa is c such that the area bounded by $x = a$, $x = b$, $y = 0$, $y = (b-c)^n f^{(n+1)}(c)$ is equal to the corresponding area bounded by $x = a$, $x = b$, $y = 0$, and the curve $y = (b-x)^n f^{(n+1)}(x)$. Hence:

$(b-a)(b-c)^n f^{(n+1)}(c) = \int_a^b (b-x)^n f^{(n+1)}(x)dx$. Hence the error $= \frac{(b-a)(b-c)^n f^{(n+1)}(c)}{\underline{|n}}$ for the n^{th} approximating-curve.

Example 1. *Use the first approximating-curve of* $y=\sin x$ *at the point* (0, 0) *to calculate* sin 2°, *and find to how many places of decimals the approximation is correct, using Lagrange's Method.*

The first approximating-curve to $y=\sin x$ at the origin is $y=x$, where x is measured in radians.

But $2^\circ=\frac{\pi}{90}$ radians $=0{\cdot}03490659$ radians.

Substituting $x=0{\cdot}03490659$ in $y=x$, we get $y=0{\cdot}03490659$ as our value for sin 2°.

But $\frac{d^2y}{dx^2}=-\sin x$.

$\therefore\ \frac{d^2y}{dx^2}$ is certainly numerically less than 1 throughout the interval 0° to 2°. Hence the error committed in using $y=x$ is certainly less than $\frac{1}{2}\left(\frac{\pi}{90}\right)^2$, *i.e.* less than 0·0005. Hence the value of y found above is correct to at least *three* places of decimals.

$$\therefore\ \sin 2^\circ=0{\cdot}035 \text{ for certain.}$$

Example 2. *Find the second approximating-curve to* $y=\sec x$ *at the point* $\left(\frac{\pi}{4}, \sqrt{2}\right)$. *Hence obtain* sec 45° 41′, *and* sec 46° *and* sec 50°. *Estimate the errors involved in each case, and hence state the number of places of decimals in each case to which your calculation is correct.*

We have
$$y=\sec x. \quad \therefore\ \frac{dy}{dx}=\sec x\tan x.$$
$$\therefore\ \frac{d^2y}{dx^2}=\sec x\tan^2 x+\sec^3 x$$
$$=2\sec^3 x-\sec x.$$
$$\therefore\ \frac{d^3y}{dx^3}=6\sec^3 x\tan x-\sec x\tan x$$
$$=\sec x\tan x\,(6\sec^2 x-1)\ \ldots\ldots\ldots\ldots(1)$$

Thus, when $x=\frac{\pi}{4}$, $y=\sqrt{2}$, $\frac{dy}{dx}=\sqrt{2}$, $\frac{d^2y}{dx^2}=3\sqrt{2}$. Hence the second approximating-curve at $\left(\frac{\pi}{4}, \sqrt{2}\right)$ is:

$$y-\sqrt{2}=\sqrt{2}\left(x-\frac{\pi}{4}\right)+\frac{3\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^2\ \ldots\ldots\ldots\ldots(2)$$

Again, sec x and tan x both increase as x increases from 0° to 90°. Thus $\frac{d^3y}{dx^3}$ in the intervals considered, viz. 49°–50°, is less than its value when $x=60°$.

Hence $\frac{d^3y}{dx^3} < 2\sqrt{3} \times 23$ by using (1), *i.e.* <80.

(1) The interval is $41' = \frac{41}{60} \times \frac{\pi}{180} = 0{\cdot}0119264$.

$$\therefore\ y = \sqrt{2} + \sqrt{2}(0{\cdot}0119264) + \frac{3\sqrt{2}}{2}(0{\cdot}0119264)^2$$
$$= 1{\cdot}4142136 + 0{\cdot}0168665 + 0{\cdot}0003017$$
$$= 1{\cdot}4313818.$$

The error is less than $\frac{80}{6}(0{\cdot}012)^3$, *i.e.* is less than $0{\cdot}000023$. Thus the result is certainly accurate to four places of decimals.

(2) The interval is $1° = \frac{\pi}{180} = 0{\cdot}0174533$.

$$\therefore\ y = \sqrt{2} + \sqrt{2}(0{\cdot}0174533) + \frac{3\sqrt{2}}{2}(0{\cdot}0174533)^2$$
$$= 1{\cdot}4142136 + 0{\cdot}0246827 + 0{\cdot}0006462$$
$$= 1{\cdot}4395425.$$

The error is less than $\frac{80}{6}(0{\cdot}018)^3$, *i.e.* is less than $0{\cdot}000078$. Thus the result is certainly accurate to three places of decimals.

(3) The interval is $5° = \frac{\ }{36} = 0{\cdot}0872665$.

$$\therefore\ y = \sqrt{2} + \sqrt{2}(0\ 0872665) + \frac{3\sqrt{2}}{2}(0{\cdot}0872665)^2$$
$$= 1{\cdot}4142136 + 0{\cdot}1234134 + 0{\cdot}0161548$$
$$= 1{\cdot}5537818.$$

The error is less than $\frac{80}{6}\ 0{\cdot}088)^3$, *i.e.* is less than $0{\cdot}0091$. Thus the result is certainly accurate to one place of decimals, and the second place cannot be more than "one out."

EXAMPLES XVIII.

1. (i) Find the equation of the tangent to the curve $y=\sin x$ (x in radians) at the point $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$.

(ii) Hence calculate the values of $\sin x$ for the following angles and fill in the following table :—

x	60° 10′	60° 23′	60° 35′	60° 48′	61°
Calculated value of $\sin x$ . . .					
Value of $\sin x$ from the Tables . .					
Number of places of decimals to which you are correct .					

2. (i) Find the equation of the tangent to the curve $y=\cos x$ (x in radians) at the point $\left(\frac{\pi}{3}, \frac{1}{2}\right)$.

(ii) Hence calculate the values of $\cos x$ for the following angles and fill in the following table :—

x	60° 17′	60° 38′	61°	62°	65°	70°
Calculated value of $\cos x$ . . .						
Value of $\cos x$ from the Tables . .						
Number of places of decimals to which you are correct .						

3. (i) Find the equation of the tangent to the curve $y=\tan x$ (x in radians) at the point $\left(\frac{\pi}{6}, \frac{1}{\sqrt{3}}\right)$.

(ii) Hence calculate the values of $\tan x$ for the following angles and fill in the following table :—

x	30° 5′	30° 18′	30° 37′	31°	32°	33°
Calculated value of $\tan x$ . . .						
Value of $\tan x$ from the Tables . .						
Number of places of decimals to which you are correct .						

4. (i) Find the equation of the tangent at the point (0, 0) to the curve $y = x\,e^x$.

(ii) Hence calculate the value of $(0{\cdot}03)\,e^{0{\cdot}03}$.

5. Find the tangent to the curve $y = \sin^{-1} x$ at the point (0, 0). Hence calculate $\sin^{-1}(0{\cdot}004)$ (the angles being measured in radians).

6. Calculate $\sin^{-1}(0{\cdot}502)$, giving your answer in radians.

7. Calculate $25^{0{\cdot}503}$.

8. Calculate $36^{0{\cdot}507}$.

9. Find the equation of the tangent to the curve $y = \log_e \cos x$ at the point $\left(\frac{\pi}{3}, \log_e 0{\cdot}5\right)$. Hence calculate the value of $\log_e \cos 60° \, 37'$.

10. Calculate the value of $\log_e \sin 60° \, 54'$.

11. Calculate the value of $\log_e \tan 45° \, 27'$.

12. Calculate the value of $\log_{10} 10{\cdot}04$.

13. Calculate by ordinary methods the square root of 37, and hence find $37^{0{\cdot}504}$ by the Taylor approximating-curve method. Use only the geometrical tangent.

14. Find the equation of the tangent at the point (0, 0) to the curve $y = \log_e(1 + 3x)$.

15. Find the equation of the tangent at the point (0, 0) to the curve $y = \log_e(1 + x^2)$.

16. Find the equation of Taylor's second approximating-curve to $y=\sin x$ at the point $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$. Hence calculate as accurately as you can $\sin 60^\circ\ 37'$ and also $\sin 63^\circ$. Find from the Tables to how many places of decimals your calculations are correct in each case.

17. Find the equation of Taylor's second approximating-curve to $y=\cot x$ at the point $\left(\frac{\pi}{4}, 1\right)$. Hence calculate $\cot 45^\circ\ 33'$ and $\cot 50^\circ$. Find from the Tables to how many places of decimals you are correct in each case.

18. Calculate Taylor's third approximating-curve at the point (0, 1) to $y=e^x$. Hence calculate $e^{0{\cdot}03}$ and $e^{-0{\cdot}04}$.

19. Calculate Taylor's third approximating-curve at the point (0, 0) to $y=\log_e(1+x)$. Hence calculate as accurately as you can $\log_e 1{\cdot}03$ and $\log_e 0{\cdot}996$.

20. We know that $16^{\frac{1}{4}}=2$. Hence calculate $16^{0{\cdot}253}$ by using Taylor's third approximating-curve.

21. Find Taylor's third approximating-curve to $y=(1+x)^{\frac{3}{4}}$ at the point (0, 1). Hence calculate $(1{\cdot}05)^{\frac{3}{4}}$ and $(0{\cdot}998)^{\frac{3}{4}}$.

22. Find the equation of the second Taylor approximating-curve at the point $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ on the curve $y=\sin x$. Draw on squared paper the graph of $y=\sin x$ and also the Taylor curve at the point $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$, on as big a scale as your paper will allow. Make any remarks that you deem fit on the closeness of the approximation, and say from your diagram throughout what range the approximation is reasonably good.

23. Find Taylor's third approximating-curve at the point (0, 0) to the curve $y=\sin^{-1} x$ (angles measured in radians). Sketch on as big a scale as you can on squared paper the graphs of $y=\sin^{-1} x$ and the above Taylor curve. Through what range of values of x is the approximation reasonably good?

24. Apply Taylor's theorem to express $\sin\left(\frac{\pi}{4}+x\right)$ in the form of a series in ascending powers of x; and show exactly how many terms must be included, in order to give a result which is correct to five places of decimals for all positive values of x not exceeding the circular measure of 2°. (Q.P.M.S.T.)

25. Write down Maclaurin's series expressing $f(x)$ in terms of ascending powers of x, and use it to determine the series for e^x.

From this series find an approximate value for e, summing the first 8 terms of the series. (Q.P.M.S.T.)

26. Prove that the value of $(0\cdot99)^{20}$ is $0\cdot82$, correct to two places of decimals. (Q.P.M.S.T.)

[*N.B.*—Use Taylor's approximating-curves for $y = x^{20}$ at the point $(1, 1)$.]

27. Calculate the value of $(0\cdot567)^{\frac{1}{6}}$ to three places of decimals. (Q.P.M.S.T.)

[*N.B.*—Use Taylor's approximating-curves for $y=(0\cdot567)^x$ at the point $(0, 1)$.]

28. Assuming that $\log_e(1+x)$ can be expanded in the form $x-\frac{x^2}{2}+\frac{x^3}{3}-$ etc., evaluate $\log_e 1\cdot01$ correct to the sixth decimal place. Hence, being given that $\log_{10} 1\cdot01$ is $0\cdot004321$, find an approximate value of $\log_{10} e$. (Q.P.M.S.T.)

29. Write down the form of Taylor's series which expresses $f(a+x)$ in a series of ascending powers of x, and derive from it the first four terms in the expansion of $\sin(\alpha+x)$.

Apply this to determine $\sin 31^\circ$ correctly to four places, being given $\sin 30^\circ=0\cdot5$, and $\cos 30^\circ=0\cdot866$. (Q.P.M.S.T.)

30. Expand $\frac{1}{\sqrt{1+x^2}}$ in ascending powers of x, and by means of this expansion obtain an approximation to the value of $\frac{1}{\sqrt{10}}$, taking three terms of the expansion. (Q.P.M.S.T.)

31. Use Maclaurin's series to show that the expansion of $\log_e(1+x)$ in terms of x is $x-\frac{x^2}{2}+\frac{x^3}{3}\cdots$

Calculate to four significant figures the value of $\log_{10} 1\cdot08$, having given that $\log_e 10=2\cdot3026$. (Q.P.M.S.T.)

32. Find the percentage error made in taking the approximate expression $(1+3x)$ instead of the true expression $(1+x)^3$ when $x=0\cdot05$. (Q.P.M.S.T.)

33. Use Taylor's series to find the expansion of $\tan x$ in ascending powers of x as far as the term containing x^5. (Q.P.M.S.T.)

34. Find the value of $(1\cdot01)^{1\cdot6}-1\cdot016$ to five significant figures. (Q.P.M.S.T.)

35. Apply Maclaurin's theorem to expand $\log_e(1+\sin x)$ in powers of x as far as x^4. (Q.P.M.S.T.)

36. Prove that correct to the term in x^3

$$(1-3x+2x^2)^{\frac{1}{2}}=1-\frac{3x}{2}-\frac{x^2}{8}-\frac{3x^3}{16}. \qquad \text{(Q.P.M.S.T.)}$$

Illustrate the above approximation by drawing on squared paper the graphs of the right- and left-hand sides and so obtaining a diagrammatic idea of how good the approximation is.

37. Plot the graph of $y=50\left(e^{\frac{x}{20}}+e^{-\frac{x}{20}}-2\right)$ from $x=0$ to $x=10$.

Rewrite the equation by expanding the exponentials as far as x^3, and plot the new graph on the same axis. (Q.P.M.S.T.)

Discuss the closeness of the above approximation.

38. Write down Maclaurin's series for the expansion of $f(x)$, and apply it to expand $e^{\tan x}$ in a series of powers of x as far as the term in x^5. (Q.P.M.S.T.)

CHAPTER XIX.

POWER SERIES.

1. The Geometrical Series.

Consider the expression :

$$y = 1 + x + x^2 + x^3 + \dots + x^n.$$

If we multiply both sides by $1 - x$ it is easy to show that

$$y = 1 + x + x^2 + x^3 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

If x be numerically less than *unity*, it is easy to see that, as n increases, x^{n+1} becomes very small. For if we proceed indefinitely taking only a fractional part of a given object, it is perfectly plain that that fractional portion will soon be very small indeed. For example, $\frac{1}{8} = 0{\cdot}125$, $\frac{1}{8^2} = 0{\cdot}015625$, $\frac{1}{8^3} = 0{\cdot}001953125$, and so on, it being easily seen that the zeros immediately after the decimal point rapidly increase in number, as we take higher powers of $\frac{1}{8}$.

If, therefore, we make n infinitely big, we see that

$$y = 1 + x + x^2 + x^3 + \dots + x^n + \dots \text{ad } \infty = \frac{1}{1 - x}.$$

We note that

$$y = 1 + x$$
$$y = 1 + x + x^2$$
$$y = 1 + x + x^2 + x^3, \text{ etc.}$$

are the first, second, third, etc., approximating-curves to the curve $y = \frac{1}{1 - x}$, within the range of values $x = -1$ to $x = +1$ (since in the above discussion x was assumed to be numerically less than unity).

We also note that the above discussion does not hold when $x = -1$ and when $x = +1$.

The subjoined diagram exhibits the curve $y = \frac{1}{1-x}$ and its first *three* approximating-curves.

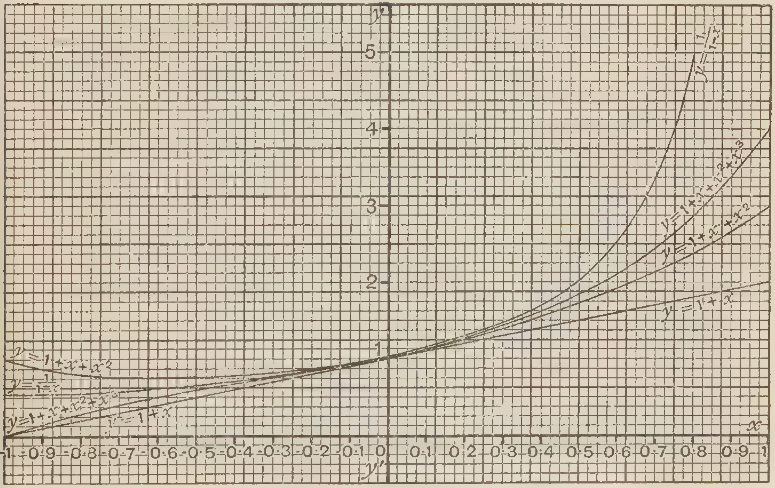

FIG. 184.

When x is positive and greater than unity, it is plain that, as n becomes infinitely large, x^n becomes indefinitely large, and hence the expression

$$y = 1 + x + x^2 + x^3 + \dots + x^n$$

becomes infinitely great as n becomes infinitely large.

When x is negative and greater than unity in numerical value $\left(e.g.\ x = -\frac{17}{3}\right)$, it is evident that x^n is very large and positive when n is very large and even, but x^n is very large and negative when n is very large and odd. Hence the expression

$$y = 1 + x + x^2 + x^3 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

is very large and positive when n is very large and even, but very large and negative when n is very large and odd.

We can see, therefore, very plainly that it is only *within* the range $x = -1$ to $x = +1$ that the curve $y = \frac{1}{1-x}$ possesses an indefinite series of approximating-curves.

RANGE OF VALIDITY.—*The range $x = -1$ to $x = +1$ is called the "Range of Validity," within which the curve $y = \frac{1}{1-x}$ has an indefinite series of "approximating-curves" of the type described above, approaching to it indefinitely close in the limit.*

2. Power Series.

The above example of the Geometric Series is merely a particular instance of Power Series in general.

Power Series are of very great utility in all branches of mathematical work.

Let $$y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n,$$

where $a_0, a_1, a_2, a_3, \dots$ are known constants, independent of x.

If, within the range $x = -h$ to $x = +h$, the expression

$$y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

approaches indefinitely close to the curve $y = f(x)$ as n becomes indefinitely increased, we regard

$$\begin{aligned} y &= a_0 + a_1x \\ y &= a_0 + a_1x + a_2x^2 \\ y &= a_0 + a_1x + a_2x^2 + a_3x^3 \\ &\text{etc.} \end{aligned}$$

as the first, second, third, etc., approximating-curves to the curve $y = f(x)$ where it crosses the axis of y. In fact, we find it convenient to regard $y = f(x)$ as the ∞^{th} "approximating-curve" and to write it:

$$y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \text{ ad } \infty \equiv f(x)(-h < x < +h).$$

Such an expression as

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \text{ ad } \infty$$

is called a "*power series*" in x.

Such an expression as

$$b_0 + b_1(x-c) + b_2(x-c)^2 + \dots + b_n(x-c)^n + \dots \text{ ad } \infty$$

is called a "*power series*" in $x - c$.

The expression

$$y = f(x) \equiv a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \text{ ad } \infty$$

regards the curve $y = f(x)$ as having an infinite series of approximating-curves where it crosses the axis of y.

The expression

$$y = f(x) \equiv b_0 + b_1(x-c) + b_2(x-c)^2 + \dots + b_n(x-c)^n + \dots \text{ ad } \infty$$

regards the curve $y = f(x)$ as having an infinite series of approximating-curves at the point on the curve $y = f(x)$ corresponding to $x = c$.

Examples of power series are:

$$\frac{1}{1} + \frac{x}{2} + \frac{x^2}{3} + \dots + \frac{x^n}{n+1} + \dots \text{ ad } \infty$$

$$1^2 + 2^2x + 3^2x^2 + \dots + (n+1)^2x^n + \dots \text{ ad } \infty$$

$$1 . 2 + 2 . 3(x-c) + 3 . 4(x-c)^2 + \dots + (n+1)(n+2)(x-c)^n + \dots \text{ ad } \infty .$$

3. Example. To find the range of validity within which $f(x) \equiv \sin x$ can be represented by a series of approximating-curves according to the method of Taylor and Maclaurin. We have, as explained in Chap. XVIII., Art. 4,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\underline{|2}}f''(0) + \dots + \frac{x^n}{\underline{|n}}f^{(n)}(0) + \frac{x^{n+1}}{\underline{|n+1}}f^{(n+1)}(c),$$

where c lies between 0 and x, and where we are considering the series of approximating-curves at the point where the curve $y = f(x)$ cuts the axis of y.

Plainly
$$\begin{array}{rlcrl} f(x) = & \sin x & \therefore & f(0) = & 0 \\ f'(x) = & \cos x & \therefore & f'(0) = & 1 \\ f''(x) = & -\sin x & \therefore & f''(0) = & 0 \\ f'''(x) = & -\cos x & \therefore & f'''(0) = & -1 \\ f^{(\text{iv})}(x) = & \sin x & \therefore & f^{(\text{iv})}(0) = & 0 \\ & \text{etc.} & & & \text{etc.} \end{array}$$

Hence $$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \ldots \pm \frac{x^n}{\underline{|n}} \mp \frac{\sin c}{\underline{|n+1}} \quad \text{.........(1)}$$

where n must be odd and the sign before each term depends on whether it is an odd or even term counting from the beginning.

Now, since $\sin c$ is always less than 1, it is plain that the error committed in stopping short at $\dfrac{x^n}{\underline{|n}}$ (n odd) is numerically less than $\dfrac{\sin c}{\underline{|n+1}}$, which becomes indefinitely small as n becomes indefinitely great, whatever be the value of c $(0 < c < x)$.

Writing (1) in the form:

$$\sin x \pm \frac{\sin c}{\underline{|n+1}} = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} \ldots \pm \frac{x^n}{\underline{|n}} \quad \text{.........(2)}$$

and proceeding to the limit on both sides of (2), as n becomes ∞ we obtain:

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \ldots \text{ad } \infty \quad \text{whatever be the value of } x,$$

i.e. the range of validity is $-\infty$ to $+\infty$.

4. Integration of Power Series.

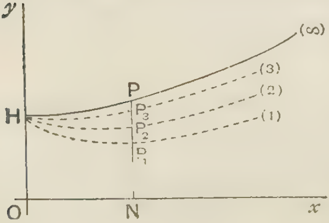

FIG. 185

Let in fig. 185 the curves marked (1), (2), (3), etc., be the first, second, third, etc., approximating-curves to the curve $y = f(x)$ marked (∞).

Let N be a point on Ox inside the range of validity. Then plainly the areas ONP$_1$H, ONP$_2$H, ONP$_3$H, etc., approximate indefinitely close to the area ONPH (where in each case the arcs of the curves act as the upper boundaries to the respective areas).

Let the given curve be

$$y=f(x)\equiv a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots\text{ad }\infty$$

and let ON $=x$.

Then the equations of the approximating-curves are respectively:

$$(1)\quad y=a_0+a_1x$$
$$(2)\quad y=a_0+a_1x+a_2x^2$$
$$(3)\quad y=a_0+a_1x+a_2x^2+a_3x^3$$
etc.
$$(\infty)\quad y=a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_nx^n+\ldots\text{ad }\infty.$$

Hence, integrating, we have:

$$(1)'\quad \text{area of ONP}_1\text{H}=a_0x+a_1\frac{x^2}{2}$$

$$(2)'\quad \text{area of ONP}_2\text{H}=a_0x+a_1\frac{x^2}{2}+a_2\frac{x^3}{3}$$

$$(3)'\quad \text{area of ONP}_3\text{H}=a_0x+a_1\frac{x^2}{2}+a_2\frac{x^3}{3}+a_3\frac{x^4}{4}$$
etc.
$$(\infty)'\quad \text{area of ONPH}=\int_0^x f(x)\,dx.$$

Hence, since the areas ONP$_1$H, ONP$_2$H, ONP$_3$H, etc., approximate indefinitely close to the area ONPH, therefore the curves

$$y=a_0x+a_1\frac{x^2}{2}$$

$$y=a_0x+a_1\frac{x^2}{2}+a_2\frac{x^3}{3}$$

$$y=a_0x+a_1\frac{x^2}{2}+a_2\frac{x^3}{3}+a_3\frac{x^4}{4}$$

approximate indefinitely close to the curve

$$y=\int_0^x f(x)\,dx,$$

or, as it is usually written:

$$\int_0^x f(x)\,dx=a_0x+a_1\frac{x^2}{2}+a_2\frac{x^3}{3}+a_3\frac{x^4}{4}+\ldots+a_n\frac{x^{n+1}}{n+1}\ldots \text{ ad } \infty$$

provided that x lies *within* the range of validity of

$$y=a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots \text{ ad } \infty .$$

Example 1. We have proved in § 1 that

$$\frac{1}{1-x}=1+x+x^2+x^3+\ldots+x^n+\ldots$$

provided that x lie *within* the range of validity -1 to $+1$.

Hence, integrating both sides from 0 to x, we get:

$$\log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+\ldots-\frac{x^{n+1}}{n+1}-\ldots \text{ ad } \infty,$$

where x must lie *within* the range of validity -1 to $+1$.

N.B.—This series is of the greatest practical utility in calculating the values of logarithms to the base e.

Example 2. By the geometrical series we know that

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\ldots+(-1)^nx^{2n}+\ldots \text{ ad } \infty \ (-1<x<+1).$$

Hence, integrating both sides from 0 to x, we obtain:

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots+(-1)^n\frac{x^{2n+1}}{2n+1}+\ldots \text{ ad } \infty \ (-1<x<+1).$$

5. Differentiation of Power Series.

In the diagram of Art. 4 it is plain that, if we draw the tangents to the respective curves at the points P_1, P_2, P_3, ... P, then the gradients of the tangents at P_1, P_2, P_3, etc., will approximate in value indefinitely close to that of the tangent at P. But gradient of tangent at $P_1 = a_1$

" " $P_2 = a_1 + 2a_2x$

" " $P_3 = a_1 + 2a_2x + 3a_3x^2$

" " $P_4 = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$
etc.

" " $P = f'(x)$.

Hence the series of curves:

$$y = a_1 + 2a_2x$$
$$y = a_1 + 2a_2x + 3a_3x^2$$
$$y = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$$
etc.

approximate indefinitely close to the curve $y = f'(x)$ provided that x lie within the range of validity of

$$y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \text{ ad } \infty,$$

or as it is usually written:

$$f'(x) \equiv a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} + \dots \text{ ad } \infty.$$

Example. We have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + \dots \text{ ad } \infty \ (-1 < x < 1).$$

Hence, differentiating both sides, we obtain:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots + nx^{n-1} + \dots \text{ ad } \infty \ (-1 < x < 1).$$

EXAMPLES XIX.

1. Given that $\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \dots + x^n$, find by differentiating both sides the value of $1 + 2x + 3x^2 + \dots + nx^{n-1}$.

2. By differentiating both sides of the result thus obtained, find the value of $1 \cdot 2 + 2 \cdot 3x + 3 \cdot 4x^2 + \dots (n-1)nx^{n-2}$.

3. Use the calculus together with the fact that

$$\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \dots + x^n$$

to find the value of $1^2 + 2^2x + 3^2x^2 + \dots + n^2x^{n-1}$.

4. Given that $\frac{1}{1-x}=1+x+x^2+x^3+\ldots+x^n+\ldots$ ad ∞, where x lies between -1 and $+1$, find the value of

$$1.2.3+2.3.4x+3.4.5x^2+\ldots+(n-2)(n-1)nx^{n-3}+\ldots \text{ ad } \infty.$$

5. Given that

$$y=1+\tfrac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\ldots+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}x^n+\ldots \text{ ad } \infty,$$

where x is numerically less than unity, prove that $(1-x)\frac{dy}{dx}-\frac{1}{2}y=0$. Hence find the value of y in terms of x.

6. Use the results of the previous question to prove that

$$\frac{1}{\sqrt{1-x^2}}=1+\frac{x^2}{2}+\frac{1.3}{2.4}x^4+\frac{1.3.5}{2.4.6}x^6+\ldots+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}x^{2n}+\ldots \text{ ad } \infty,$$

where x is numerically less than unity.

7. Use question 6 to obtain a power series whose value is $\sin^{-1}x$, valid within the range $x=-1$ to $x=+1$.

8. Given that $y=1+nx+\frac{n(n-1)}{\underline{|2}}x^2+\frac{n(n-1)(n-2)}{\underline{|3}}x^2+\ldots$

$$\ldots+\frac{n(n-1)\ldots(n-r+1)}{\underline{|r}}x^r+\ldots \text{ ad } \infty,$$

for values of x lying within the range $x=-1$ to $x=+1$, prove that $(1+x)\frac{dy}{dx}=ny$. Hence obtain an expression for y in terms of x.

9. Given that $y=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots+\frac{x^n}{\underline{|n}}+\ldots$ ad ∞, for all values of x, find $\frac{dy}{dx}$ in terms of y, and thence obtain a simple expression for y in terms of x.

10. Expand $\frac{x}{1+x^2}$ as a power series in x, integrate both sides, and hence find a power series for $\log_e\sqrt{1+x^2}$.

11. Expand $\frac{x^2}{1+x^3}$ as a power series in x, integrate both sides, and hence find a power series for $\log_e\sqrt[3]{1+x^3}$. State within what range of values for x the expansion is valid.

12. Express $(1-x^2)^{-1}$ as a power series in x, integrate both sides, and hence find an expression for another power series.

13. Given that

$$y=1-\tfrac{1}{2}x^2-\frac{1}{2.4}x^4-\frac{1.3}{2.4.6}x^6-\ldots-\frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)2n}x^{2n}-\ldots\text{ad }\infty,$$

within the range $x=-1$ to $x=+1$, find the value of $\frac{1}{x}\frac{dy}{dx}$ as a simple function. Hence obtain a simple expression for y in terms of x.

14. Given that $y=x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}-\frac{x^7}{\lfloor 7}+\ldots+(-1)^{n-1}\frac{x^{2n-1}}{\lfloor 2n-1}+\ldots\text{ad }\infty$, for all values of x, prove that $\frac{d^2y}{dx^2}+y=0$. Hence obtain a simple expression for y in terms of x.

15. Given that $y=1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\frac{x^6}{\lfloor 6}+\ldots+(-1)^n\frac{x^{2n}}{\lfloor 2n}+\ldots\text{ad }\infty$, for all values of x, prove that $\frac{d^2y}{dx^2}+y=0$. Hence obtain a simple expression for y in terms of x.

CHAPTER XX.

DIFFERENTIAL EQUATIONS.

1. A differential equation involving x, y, $\frac{dy}{dx}$, but nothing higher such as $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, etc., is called a differential equation *of the first order.*

A differential equation of the first order in $\frac{dy}{dx}$ *represents geometrically a singly infinite family of curves.*

Consider the differential equation of the first order

$$\phi\left(x,\ y,\ \frac{dy}{dx}\right)=0.$$

Choose a point $P_0 \equiv (x_0,\ y_0)$ on the squared paper, and solve for $\frac{dy}{dx}$ from the above equation. We get one or more values of $\frac{dy}{dx}$. Choosing any one of these values, we draw a short line from $(x_0,\ y_0)$, whose gradient is equal to the value of $\frac{dy}{dx}$ above obtained.

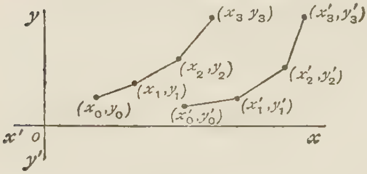

FIG. 186.

Let us proceed a very short distance from $P_0 \equiv (x_0,\ y_0)$ to $P_1 \equiv (x_1,\ y_1)$ along this line. Then, using the differential equation once more, we find that value of $\frac{dy}{dx}$ corresponding to the point $(x_1,\ y_1)$, which differs very slightly from the gradient of the short chord P_0P_1. We then draw in the direction thus obtained the short chord P_1P_2. Inserting $(x_2,\ y_2)$ in the differential equation once more, we find the corresponding value of

$\frac{dy}{dx}$, which differs very slightly from the gradient of the chord P_1P_2. We draw the very short chord P_2P_3 parallel to the direction thus obtained, and so on. In the limit, the series of short chords P_0P_1, P_1P_2, P_2P_3, etc., become a continuous curve.

In the same way, if we start from any other point $P_0' \equiv (x_0', y_0')$, we obtain a curve which is the limiting position of the series of short arcs $P_0'P_1'$, $P_1'P_2'$, $P_2'P_3'$, etc.

Hence the differential equation of the first order is represented graphically by a singly-infinite series of curves.

Example. *Find the series of curves that represents graphically the differential equation* $\frac{dy}{dx} = 6x$.

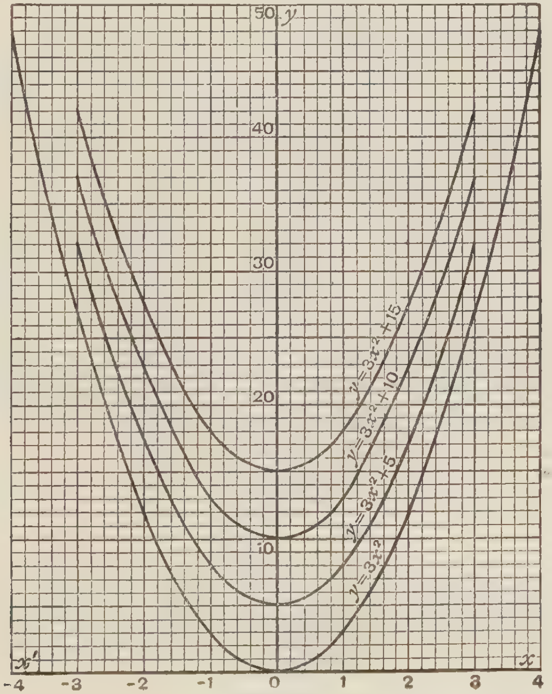

FIG. 187.

We have
$$dy = 6x\,dx.$$
$$\therefore\ y = 3x^2 + C.$$

Fig. 187 exhibits some members of this family.

2. To solve differential equations of the type

$$\frac{dy}{dx} + Py = Q,$$

where P and Q are functions of x only.

Multiply the given differential equation all through by u, where as yet u has not been determined.

We obtain: $$u\frac{dy}{dx} + uPy = uQ,$$

which can be written: $$\frac{d}{dx}(uy) + \left(uP - \frac{du}{dx}\right)y = uQ.$$

Let us now choose u so that the coefficient of y shall vanish, *i.e.* $uP - \frac{du}{dx} = 0$. The equation then becomes $\frac{d}{dx}(uy) = uQ$.

Hence $\frac{du}{u} = P\,dx$, which can at once be integrated as soon as P is known.

Thus the value of u in terms of x is found, and on being inserted in $\frac{d}{dx}(uy) = uQ$, the integration of the given equation can be effected immediately.

N.B.—The simplest value of u that will satisfy this equation ought to be chosen.

Example 1. *Solve the differential equation* $\frac{dy}{dx} + y = x$.

We have
$$u\frac{dy}{dx} + uy = ux,$$
$$i.e.\ \frac{d}{dx}(uy) + y\left(u - \frac{du}{dx}\right) = ux \qquad \text{.....................(1)}$$

Choose u so that $u - \frac{du}{dx} = 0$, *i.e.* $\frac{du}{u} = dx$, *i.e.* $\log_e u = x$. Hence $u = e^x$ is the simplest value of u that can be chosen.....................(2)

Hence $$\frac{d}{dx}(ye^x) = xe^x \text{ by (1) and (2).}$$

$$\therefore\ ye^x = \mathrm{C} + \int xe^x\,dx.$$

$$\therefore\ ye^x = \mathrm{C} + xe^x - e^x,$$

$$i.e.\ y = (x-1) + \mathrm{C}e^{-x}.$$

The subjoined diagram exhibits graphically some members of the family of curves defined by the given differential equation.

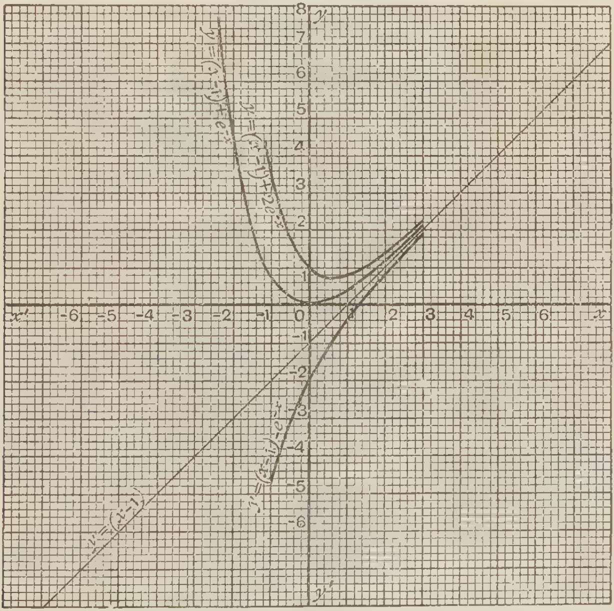

FIG. 188.

Example 2. *The following equation occurs in dealing with electrical circuits,* L *being the coefficient of self-induction,* R *the resistance,* E *the external electromotive force in the circuit :*

$$\mathrm{L}\frac{di}{dt} + \mathrm{R}i = \mathrm{E}.$$

Solve for the current i, *it being given that* L, R, E *are known constants.*

We have
$$\frac{di}{dt} + \frac{\mathrm{R}}{\mathrm{L}}i = \frac{\mathrm{E}}{\mathrm{L}}.$$

Hence $$u\frac{di}{dt}+\frac{R}{L}ui=\frac{E}{L}u.$$

$$\therefore \frac{d(ui)}{dt}+i\left(\frac{R}{L}u-\frac{du}{dt}\right)=\frac{E}{L}u \quad\text{.............................}(1)$$

Let $$\frac{R}{L}u-\frac{du}{dt}=0.$$

$$\therefore \frac{du}{u}=\frac{R}{L}dt.$$

$$\therefore \log_e u=\frac{R}{L}t.$$

$$\therefore u=e^{\frac{R}{L}t} \quad\text{.............................}(2)$$

Hence $$\frac{d}{dt}\left(ie^{\frac{R}{L}t}\right)=\frac{E}{L}e^{\frac{R}{L}t} \text{ by (1) and (2).}$$

$$\therefore ie^{\frac{R}{L}t}=C+\frac{E}{R}e^{\frac{R}{L}t}.$$

$$i.e.\ i=\frac{E}{R}+Ce^{-\frac{R}{L}t}.$$

3. To solve differential equations of the type

$$\mathbf{\frac{dy}{dx}=\frac{ax+by+c}{a'x+b'y+c'}.}$$

Transfer the origin of coordinates to the point of intersection of the lines:

$$ax+by+c=0$$
$$a'x+b'y+c'=0,$$

i.e. to the point $\left(\frac{bc'-b'c}{ab'-a'b}, \frac{ca'-c'a}{ab'-a'b}\right)$.

Let $$x=X+\frac{bc'-b'c}{ab'-a'b}$$

and $$y=Y+\frac{ca'-c'a}{ab'-a'b}.$$

Hence, substituting in the given equation, we obtain:

$$\frac{dY}{dX}=\frac{aX+bY}{a'X+b'Y} \quad\text{.........................}(1)$$

To solve this equation, put $Y = VX$ ....(2)

$$\therefore \frac{dY}{dX} = X\frac{dV}{dX} + V \qquad\text{........................(3)}$$

Hence, by (1), (2), and (3),

$$X\frac{dV}{dX} + V = \frac{a+bV}{a'+b'V}.$$

$$\therefore X\frac{dV}{dX} = \frac{a+(b-a')V-b'V^2}{a'+b'V}.$$

$$\therefore \frac{(a'+b'V)dV}{a+(b-a')V-b'V^2} = \frac{dX}{X}.$$

Hence the variables are separated and the equation can be integrated.

Example. *Solve the differential equation:*

$$\frac{dy}{dx} = \frac{2x-y-1}{x+3y-11}.$$

The two lines $2x - y - 1 = 0$

and $x + 3y - 11 = 0$

intersect in the point (2, 3). We transfer to parallel axes through this point as origin by putting

$$x = X + 2,$$

$$y = Y + 3.$$

Hence $$\frac{dY}{dX} = \frac{2X-Y}{X+3Y}.$$

Hence, substituting $Y = VX$, we obtain:

$$X\frac{dV}{dX} + V = \frac{2-V}{1+3V}.$$

$$\therefore X\frac{dV}{dX} = \frac{2-2V-3V^2}{1+3V}.$$

$$\therefore \frac{(1+3V)dV}{2-2V-3V^2} = \frac{dX}{X},$$

$$\text{i.e. } \frac{(\frac{1}{3}+V)dV}{\frac{2}{3}-\frac{2}{3}V-V^2} = \frac{dX}{X}.$$

Hence, "completing the square" in case of the quadratic expression in V, we obtain:

$$\frac{(\frac{1}{3}+V)dV}{\frac{7}{9}-(V+\frac{1}{3})^2}=\frac{dX}{X},$$

$$\textit{i.e.}\ -\frac{\frac{1}{2}d(V+\frac{1}{3})^2}{(V+\frac{1}{3})^2-\frac{7}{9}}=\frac{dX}{X},$$

$$\textit{i.e.}\ -\frac{1}{2}\,\frac{d\{(V+\frac{1}{3})^2-\frac{7}{9}\}}{(V+\frac{1}{3})^2-\frac{7}{9}}=\frac{dX}{X}.$$

$$\therefore\ \tfrac{1}{2}\log C-\tfrac{1}{2}\log\{(V+\tfrac{1}{3})^2-\tfrac{7}{9}\}=\log X,$$

$$\textit{i.e.}\ \tfrac{1}{2}\log\frac{C}{(V+\frac{1}{3})^2-\frac{7}{9}}=\log X.$$

$$\therefore\ \frac{C}{(V+\frac{1}{3})^2-\frac{7}{9}}=X^2.\qquad\therefore\ 9C=(3Y+X)^2-7X^2.$$

Several members of this family of curves are shown below in fig. 189.

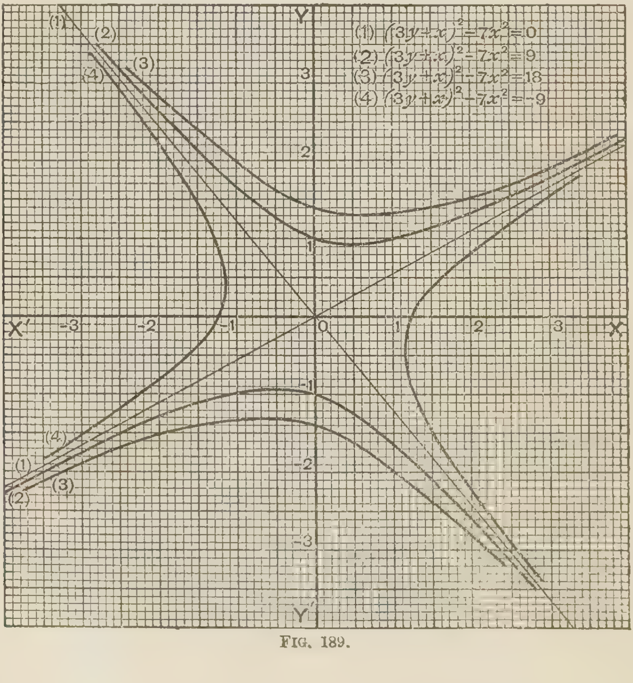

FIG. 189.

I. *Geometrical aspect of the system of curves representing the differential equation* $\dfrac{dY}{dX}=\dfrac{aX+bY}{a'X+b'Y}$.

The equation can be written $\dfrac{dY}{dX}=\dfrac{a+b\dfrac{Y}{X}}{a'+b'\dfrac{Y}{X}}$.

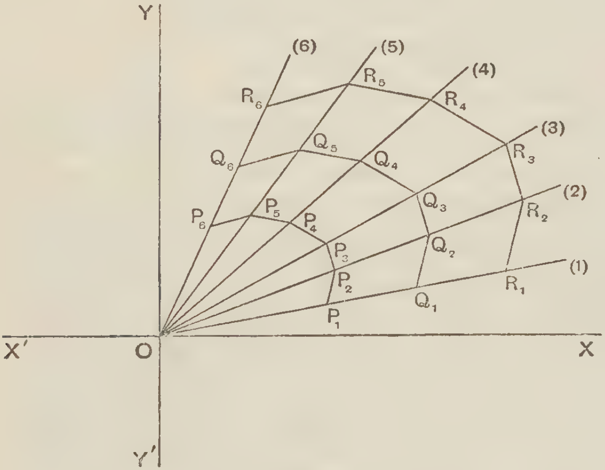

FIG. 190.

Consider the line (1). $\dfrac{dY}{dX}$ is the gradient of any curve of the system at the point (X, Y), and hence all the curves of the system must have the same gradient at points of the line (1) since $\dfrac{Y}{X}$ is constant for all points of the line (1).

Starting from points on the line (1), we therefore draw a series of short parallel lines to meet the line (2) (it being supposed that (2) is very near (1)), these short parallel lines representing the gradients of the tangents to the curves of the system where they cut the line (1).

Reasoning as above, we see that similarly all the curves of the system must have the same gradient at the points where they cut the line (2). We therefore draw a series of parallel lines representing the gradients of the curves of the system as they cross the line (2).

But in the limit $P_1P_2P_3P_4P_5P_6$, $Q_1Q_2Q_3Q_4Q_5Q_6$, $R_1R_2R_3R_4R_5R_6$, etc., are curves of the system, and they possess the property of being "*similar and similarly situated curves*" according to the geometrical definition, the origin O being the Centre of Similarity. For if one of the curves of the system be given we can obtain all the others thus:

Let P be a point on the given curve. Join O to P and produce OP to Q so that OP : OQ is constant. Then if we do this for a number of points P, we obtain a number of points Q on the new curve, which can then be sketched. The figure to the example of this article shows graphically the above property.

II. *All the flexes (i.e. points of inflexion) of the family of curves represented by the differential equation* $\frac{dY}{dX}=\frac{aX+bY}{a'X+b'Y}$ *lie on one or other of two straight lines passing through the origin.*

At a point of inflexion on a curve $\frac{d^2Y}{dX^2}$ is equal to zero.

Hence, differentiating the given equation, we obtain:

$$\frac{d^2Y}{dX^2}=\frac{(a'X+b'Y)\left(a+b\frac{dY}{dX}\right)-(aX+bY)\left(a'+b'\frac{dY}{dX}\right)}{(a'X+b'Y)^2}$$

$$=\frac{(ab'-a'b)\left(Y-X\frac{dY}{dX}\right)}{(a'X+b'Y)^2}.$$

If, therefore, $\frac{d^2Y}{dX^2}$ vanishes, we have $Y-X\frac{dY}{dX}=0.$

Hence, eliminating $\frac{dY}{dX}$ between this result and the given equation, we have $$\frac{Y}{X}=\frac{aX+bY}{a'X+b'Y}.$$

The pair of lines required are therefore:

$$aX^2+(b-a')XY-b'Y^2=0.$$

N.B.—Since $\dfrac{dY}{dX} = \dfrac{Y}{X}$ at a flex, we note that the gradient of the curve at a flex is equal to the gradient of the line joining the origin to that flex, *i.e.* the tangent at a flex passes through the origin.

4. Useful principle.

The following represents a very useful principle, especially in dealing with the graphical solution of differential equations :—

If $\phi\left(x, y, \dfrac{dy}{dx}\right) = 0$ *be the differential equation under investigation, then* $\phi(x, y, \mathrm{C}) = 0$ *represents the locus of points at which the gradients of the curves of the system represented by the differential equation are each* C.

The subjoined diagram will make this plain. Consider the

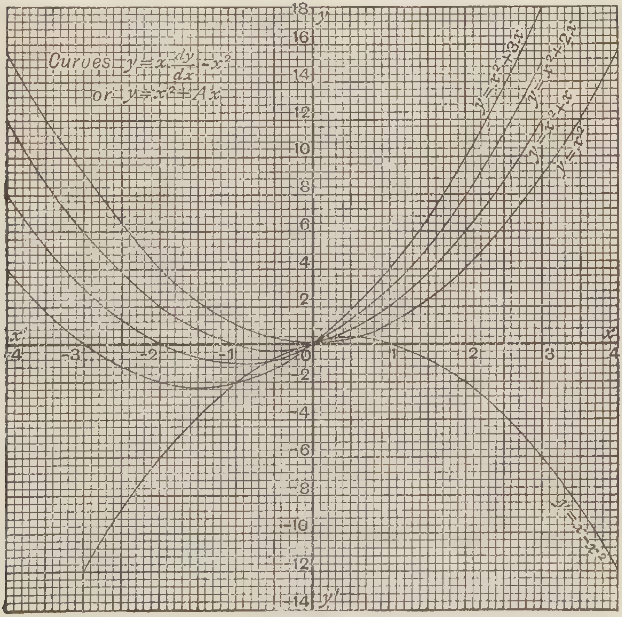

FIG. 191.

differential equation $y = x\dfrac{dy}{dx} - x^2$. The locus of points at which the gradients of the respective curves through them is 1 is $y = x - x^2$ $\left(\text{putting } \dfrac{dy}{dx} = 1 \text{ in the given differential equation}\right)$.

5. Clairaut's Differential Equations.

To solve differential equations of the type $y = x\dfrac{dy}{dx} + f\left(\dfrac{dy}{dx}\right)$.

METHOD (1). *By Geometry.*

The straight line $y = Cx + f(C)$ represents the locus of points, the tangents to the curves of the family through which have all gradients equal to C.

Consider the points P, Q, R, S, T, etc., lying near one another on the line $y = Cx + f(C)$. Let us draw, by the method

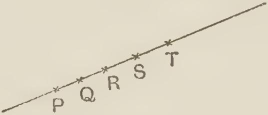

FIG. 192.

of Art. 1 of the present chapter, the curve of the system passing through P. The gradient of the tangent to the curve through P is C. But the gradient of the above line is C, as is easily seen from its equation. Hence the short line PQ is a very short arc of the curve through P. Reasoning in the same way, we see that PQ, QR, RS, ST, etc., are all short arcs of this curve. In fact, the curve through P is simply the line $y = Cx + f(C)$.

Hence the family of curves satisfying the differential equation $y = x\dfrac{dy}{dx} + f\left(\dfrac{dy}{dx}\right)$ *is the family of straight lines* $y = Cx + f(C)$.

Corollary. *If the above family of straight lines all touch the same curve* S_0, *this latter curve is also a solution of the given differential equation.*

For let the line l be one of the lines of the system, and let it touch the annexed curve S_0 at the point L. Let $L \equiv (x, y)$, and let the gradient of the line l be C. Then plainly $x, y, \frac{dy}{dx} = C$

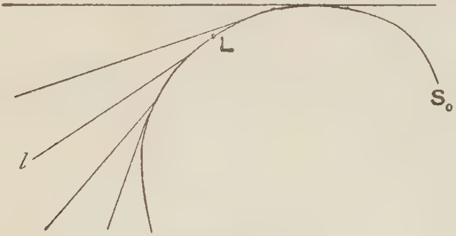

FIG. 193.

satisfies the given differential equation. Hence the point L and the gradient $\frac{dy}{dx}$ of the tangent to the curve S_0 at L satisfy the given differential equation. Thus the equation to the curve S_0 is a solution of the given differential equation.

Note 1.—S_0 is not a member of the system of curves $y = Cx + f(C)$, which are straight lines.

Note 2.—S_0 is called a "*Singular Solution*," since it is not a member of the family of curves that are usually considered to be the solution of the given differential equation.

METHOD (2). *By Analysis.*

Consider
$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right) \text{......................} \quad (1)$$

Differentiating to x, we obtain:

$$\frac{dy}{dx} = x\frac{d^2y}{dx^2} + \frac{dy}{dx} + f'\left(\frac{dy}{dx}\right) \times \frac{d^2y}{dx^2}.$$

$$\therefore \left\{x + f'\left(\frac{dy}{dx}\right)\right\}\frac{d^2y}{dx^2} = 0.$$

Hence, either
$$\frac{d^2y}{dx^2} = 0, \text{ i.e. } \frac{dy}{dx} = C \text{......................} (2)$$

or
$$x + f'\left(\frac{dy}{dx}\right) = 0. \text{........................} (3)$$

From (1) and (2) we obtain the general solution

$$y = Cx + f(C).$$

From (1) and (3) we obtain the Singular Solution.

6. Graphical solution of differential equations.

Just as, in ordinary equations, it is often impossible to solve them algebraically, in which case an approximate solution has to be obtained by means of squared paper, so in the case of differential equations, we have often to obtain an approximate solution graphically.

Consider the differential equation $\phi\left(x, y, \frac{dy}{dx}\right) = 0$.

Consider also the system of curves $\phi(x, y, C) = 0$ obtained by giving different values to C.

As explained in Art. 4, the curves of the family that satisfy $\phi\left(x, y, \frac{dy}{dx}\right) = 0$ each have the same gradient C at points where they cut $\phi(x, y, C) = 0$.

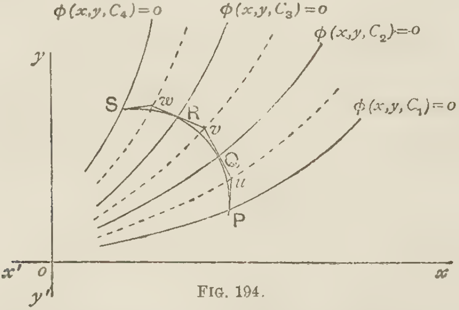

FIG. 194.

Let the thick curves in the above diagram be several members of the family $\phi(x, y, C) = 0$ not very far apart (*i.e.* let C_1, C_2, C_3, C_4, etc., be values of C differing by a small amount). Let the dotted curves be sketched so as to lie as nearly as possible half-way between the thick curves. Take any point P on $\phi(x, y, C_1) = 0$ and draw a short line Pu whose gradient is C_1. From u draw a short line uQv whose gradient is C_2. From v draw a short line vRw whose gradient is C_3. From w draw a short line wS whose gradient is C_4, and so on. Then sketch in as carefully as possible a curve touching Pu at P, uQv at Q, vRw at R, wS at S, and so on. The curve PQRS thus obtained is very approximately the curve through P satisfying the differential equation $\phi\left(x,\ y,\ \frac{dy}{dx}\right)=0$.

Example. *Solve graphically the differential equation* $y=x\frac{dy}{dx}-x^2$, *and obtain approximately the curve of the system that passes through the point* (2, 3).

Putting $x=2$, $y=3$ in $y=Cx-x^2$, we find that $C=3{\cdot}5$.

(1) $y=3{\cdot}5x-x^2$, (2) $y=3{\cdot}7x-x^2$, (3) $y=3{\cdot}9x-x^2$, (4) $y=4{\cdot}1x-x^2$.
(α) $y=3{\cdot}6x-x^2$, (β) $y=3{\cdot}8x-x^2$, (γ) $y=4x-x^2$.

Gradient of P$u=3{\cdot}5$, gradient of uQ$v=3{\cdot}7$.
Gradient of vR$w=3{\cdot}9$, gradient of wS$=4{\cdot}1$.

Thus PQRS, practically a continuous curve, is the graphical solution of $y=x\frac{dy}{dx}-x^2$.

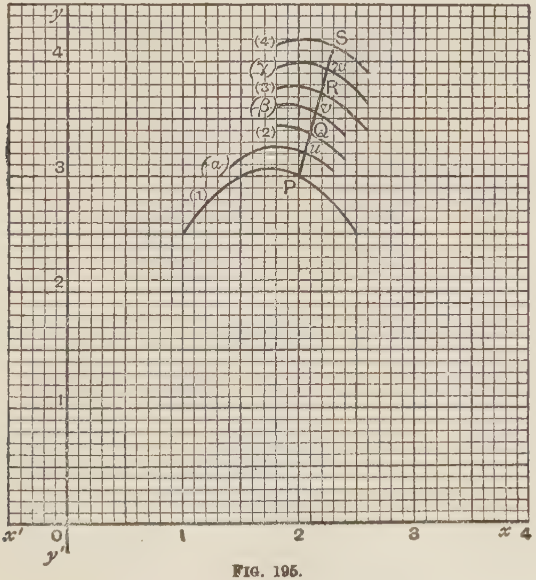

FIG. 195.

7. Differential Equations and Approximating-Series.

The method of approximating-series can also be applied to finding approximate solutions to given differential equations. We give some examples.

Example 1. *Given the differential equation* $x\frac{dy}{dx}=y+x^2+y^2$, *find the third approximating-curve to the curve of the system (as defined by the above differential equation) that passes through the point* (1, 1).

The third approximating-curve is

$$(y-1)=\left(\frac{dy}{dx}\right)_1(x-1)+\tfrac{1}{2}\left(\frac{d^2y}{dx^2}\right)_1(x-1)^2+\tfrac{1}{6}\left(\frac{d^3y}{dx^3}\right)_1(x-1)^3 \text{.........(1)}$$

where $\left(\frac{dy}{dx}\right)_1$, $\left(\frac{d^2y}{dx^2}\right)_1$, $\left(\frac{d^3y}{dx^3}\right)_1$ are the values of $\left(\frac{dy}{dx}\right)$, $\left(\frac{d^2y}{dx^2}\right)$, $\left(\frac{d^3y}{dx^3}\right)$ respectively at the point (1, 1).

We have $$x\frac{dy}{dx}=y+x^2+y^2 \text{.................................(2)}$$

Differentiating (2), we obtain :

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{dy}{dx}+2x+2y\frac{dy}{dx},$$

$$i.e.\ \ x\frac{d^2y}{dx^2}=2x+2y\frac{dy}{dx} \text{.............................(3)}$$

Also differentiating (3), again we obtain :

$$x\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}=2+2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2 \text{.......................(4)}$$

From (2), putting $x=1$ and $y=1$, we obtain :

$$\left(\frac{dy}{dx}\right)_1=3.$$

From (3), putting $x=1, y=1, \frac{dy}{dx}=3$, we get :

$$\left(\frac{d^2y}{dx^2}\right)_1=8,$$

and from (4): $$\left(\frac{d^3y}{dx^3}\right)_1=28.$$

Hence the equation to the *third* approximating-curve at the point (1, 1) is

$$(y-1)=3(x-1)+4(x-1)^2+\tfrac{14}{3}(x-1)^3.$$

To find the value of y when $x=1\cdot 03$, we insert for x in the above and obtain :

$$y-1=3(0\cdot 03)+4(0\cdot 03)^2+\tfrac{14}{3}(0\cdot 03)^3.$$

$$\therefore\ y=1\cdot 094 \text{ (to 3 places of decimals).}$$

Example 2. *Show that of the system of curves defined by the differential equation*

$$(2x+1)\left(\frac{dy}{dx}\right)^2-10x\frac{dy}{dx}+(x+y+4)=0$$

two pass through any given point.

Find the first approximating-curves to the two curves of the system passing through the points (0, 0) *and* (1, 1).

If we assign values to x and y and treat the differential equation thereafter as a quadratic in $\frac{dy}{dx}$, we obtain *two* values of $\frac{dy}{dx}$ corresponding to a given point (x, y). Hence *two* curves of the system pass through any given point.

I. The point (0, 0).

If we put $x=0$ and $y=0$ in the equation

$$(2x+1)\left(\frac{dy}{dx}\right)^2-10x\frac{dy}{dx}+(6x+y-4)=0$$

we obtain $$\left(\frac{dy}{dx}\right)=4.$$

$$\therefore\ \frac{dy}{dx}=+2 \text{ or } -2.$$

Hence the first approximating-curves to the two curves of the system defined by the given differential equation (passing through the point (0, 0)) are

$$y=2x \quad\text{and}\quad y=-2x.$$

II. The point (1, 1).

Putting $x=1$ and $y=1$ in the given differential equation, we obtain :

$$3\left(\frac{dy}{dx}\right)^2-10\frac{dy}{dx}+3=0. \qquad \therefore\ \frac{dy}{dx}=3 \text{ or } 0\cdot 33.$$

Hence the first approximating-curves through the point (1, 1) are

$$y-1=3(x-1)$$

and

$$y-1=0{\cdot}33(x-1).$$

Example 3. *Find the second approximating-curves to the differential equation of Example 2 through the points* (0, 0) *and* (1, 1).

Differentiating the differential equation, we get:

$$2\left(\frac{dy}{dx}\right)^2+2(2x+1)\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}-10\frac{dy}{dx}-10x\frac{d^2y}{dx^2}+6+\frac{dy}{dx}=0.$$

When $x=0\ \ y=0\ \ \frac{dy}{dx}=2$ then $\frac{d^2y}{dx^2}=1$

$x=0\ \ y=0\ \ \frac{dy}{dx}=-2$ „ $\frac{d^2y}{dx^2}=8$

$x=1\ \ y=1\ \ \frac{dy}{dx}=3$ „ $\frac{d^2y}{dx^2}=0{\cdot}38$

$x=1\ \ y=1\ \ \frac{dy}{dx}=0{\cdot}33$ „ $\frac{d^2y}{dx^2}=0{\cdot}40.$

Thus the second approximating-curves are:

$$\left.\begin{array}{l}(1)\ y=2x+0{\cdot}5x^2\\(2)\ y=-2x+4x^2\end{array}\right\}\text{through }(0,0).$$

$$\left.\begin{array}{l}(3)\ y-1=3(x-1)+0{\cdot}19(x-1)^2\\(4)\ y-1=0{\cdot}33(x-1)+0{\cdot}2(x-1)^2\end{array}\right\}\text{through }(1,1).$$

Example 4. *Using the results of Example 3, calculate the values of* y *corresponding to* $x=0{\cdot}05$ *in the case of the curves that pass through* (0, 0).

Calculate also the values of y *corresponding to* $x=1{\cdot}02$ *in the case of the curves that pass through* (1, 1).

Give your results to two places of decimals.

(1) $y=2(0{\cdot}05)+0{\cdot}5(0{\cdot}05)^2=0{\cdot}10$ to 2 places.

(2) $y=-2(0{\cdot}05)+4(0{\cdot}05)^2=-0{\cdot}09$ „ „

(3) $y=1+3(0{\cdot}02)+0{\cdot}19(0{\cdot}02)^2=1{\cdot}06$ „ „

(4) $y=1+0{\cdot}33(0{\cdot}02)+0{\cdot}2(0{\cdot}02)^2=1{\cdot}01$ „ „

8. Differential Equations of the Second Order.

A differential equation of the *first* order is an equation between x, y, $\frac{dy}{dx}$.

A differential equation of the *second* order is an equation between x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$.

A differential equation of the *third* order is an equation between x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$.

And so on.

We have seen that the general solution to a differential equation of the *first* order is an equation between x, y and *one* arbitrary constant C.

It will be seen that the general solution to a differential equation of the *second* order is an equation between x, y and *two* arbitrary constants A and B.

More generally, the general solution to a differential equation of the *third* order is an equation between x, y and *three* arbitrary constants A, B, C.

Example. *Solve* $\frac{d^3y}{dx^3}=x$, *a differential equation of the third order.*

We have
$$\frac{d^2y}{dx^2}=A+\frac{x^2}{2}$$

and
$$\frac{dy}{dx}=B+Ax+\frac{x^3}{6}.$$

Finally,
$$y=C+Bx+A\frac{x^2}{2}+\frac{x^4}{24},$$

a solution which contains *three* arbitrary constants A, B, C.

We can explain this result as follows :—

Consider
$$y=Cx^2+x \quad\text{...(1)}$$

We wish to eliminate C.

Differentiate *once*, obtaining :
$$\frac{dy}{dx}=2Cx+1 \quad\text{....................................(2)}$$

Hence, eliminating C between (1) and (2), we get :
$$y=\frac{x}{2}\left(\frac{dy}{dx}-1\right)+x,$$
$$i.e.\ y=\frac{x}{2}\frac{dy}{dx}+\frac{x}{2}.$$

Hence, to eliminate *one* arbitrary constant C, we need only differentiate *once*; that is, our final result without C will be a differential equation of the *first* order.

Again, consider $$y = Ax^2 + Bx \quad \text{.........} (1)$$

We wish to eliminate A and B.

Differentiate *once*:

$$\frac{dy}{dx} = 2Ax + B \quad \text{.........} (2)$$

Differentiate twice:

$$\frac{d^2y}{dx^2} = 2A \quad \text{.........} (3)$$

Hence, eliminating A and B between (1), (2), (3), we obtain:

$$y = x\frac{dy}{dx} - \frac{x^2}{2}\frac{d^2y}{dx^2}.$$

Hence, to eliminate *two* arbitrary constants A and B, we have to differentiate *twice*; that is, our final result without A and B will be a differential equation of the second order.

9. To solve $\frac{d^2x}{dt^2} + n^2x = 0$.

We have already solved this equation in Chap. XIII. Art. 5, and explained its relationship to vibratory motion. We shall now deal with it as an example of the principles of the last article.

Consider $$x = A \sin nt + B \cos nt.$$

It is easy to see that this satisfies the above differential equation, and as it contains *two* arbitrary constants A and B, it will be the general solution.

Corollary. *To solve* $\frac{d^2x}{dt^2} + n^2x = n^2c.$

We have $$\frac{d^2x}{dt^2} + n^2(x - c) = 0.$$

$$\therefore \frac{d^2}{dt^2}(x - c) + n^2(x - c) = 0.$$

Hence the complete solution will be:

$$x - c = A \sin nt + B \cos nt.$$

To solve $\frac{d^2x}{dt^2} - n^2x = 0.$

The general solution is plainly:

$$x = \mathrm{A}e^{nt} + \mathrm{B}e^{-nt}.$$

To solve $\frac{d^2x}{dt^2} - n^2x = n^2c.$

Writing equation $\frac{d^2(x+c)}{dt^2} - n^2(x+c) = 0$, we see that

$$x + c = \mathrm{A}e^{nt} + \mathrm{B}e^{-nt}.$$

10. Linear Differential Equations with Constant Coefficients.

A linear differential equation with constant coefficients is one of the type:

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_ny = f(x),$$

where a_0, a_1, a_2, … a_n are known constants.

We shall first of all show how to solve such equations when the right-hand side is zero, viz.:

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_ny = 0.$$

Let us take as a trial solution $y = e^{mx}$, substitute in the given equation, and cancel out e^{mx}.

We obtain:

$$a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \ldots + a_n = 0.$$

Suppose in the first instance that all the roots of this last equation are real and different from one another, and let them be m_1, m_2, m_3, … m_n. Then e^{m_1x}, e^{m_2x}, e^{m_3x}, … e^{m_nx} will each be solutions of the given differential equation, there being n solutions in all. The complete solution will therefore be:

$$y = \mathrm{A}_1e^{m_1x} + \mathrm{A}_2e^{m_2x} + \mathrm{A}_3e^{m_3x} + \ldots + \mathrm{A}_ne^{m_nx},$$

where A_1, A_2, A_3, … A_n are n arbitrary constants, and these will be sufficient since the degree of the equation is n.

Example. *Solve the equation* $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$.

Assume as a trial solution $y=e^{mx}$, and substitute in the given equation. We obtain, after cancelling e^{mx},

$$m^2-5m+6=0.$$

Hence $m=2$ or 3. Thus e^{2x} and e^{3x} are each solutions, and the general solution will be $y=\text{A}e^{2x}+\text{B}e^{3x}$, since the given differential equation is of the second degree, and therefore there will be two arbitrary constants in the general solution.

Repeated Roots for the equation in m.

Let us now consider the case when all the roots of the equation in m are real but some are repeated. In that case we should not have n independent solutions for the case of a differential equation of degree n.

We find, however, that if $m=m'$ is a root of the equation in m repeated *twice* over, then $e^{m'x}$ and $xe^{m'x}$ are *each* solutions of the given differential equation, and the part of the final solution corresponding to the twice-repeated root m' is

$$\text{A}e^{m'x}+\text{B}xe^{m'x}.$$

Similarly, if m' is a thrice-repeated root of the equation in m, then $e^{m'x}$, $xe^{m'x}$, $x^2e^{m'x}$ are each solutions of the given differential equation, and hence the part of the final solution corresponding to the thrice-repeated root m' is

$$\text{A}e^{m'x}+\text{B}xe^{m'x}+\text{C}x^2e^{m'x}.$$

And so on, for roots repeated more often.

This principle is easily proved as follows :—

Let the given differential equation be :

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_ny=0\ldots\ldots\ldots\ldots(1)$$

which may be written :

$$\left\{a_0\left(\frac{d}{dx}\right)^n+a_1\left(\frac{d}{dx}\right)^{n-1}+a_2\left(\frac{d}{dx}\right)^{n-2}+\ldots+a_n\right\}y=0\ldots\ldots(2)$$

If $y=e^{mx}$ be taken as a trial solution, the corresponding equation in m is :

$$a_0m^n+a_1m^{n-1}+a_2m^{n-2}+\ldots+a_n=0\ldots\ldots\ldots\ldots\ldots(3)$$

If the roots of (3) be $m_1, m_2, m_3, \ldots m_n$, then the equation (3) may be written:

$$a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \ldots + a_n \equiv a_0(m - m_1)(m - m_2)(m - m_3)\ldots(m - m_n).$$

Hence, changing m into $\frac{d}{dx}$, we see that (2) may be written:

$$a_0\left(\frac{d}{dx} - m_1\right)\left(\frac{d}{dx} - m_2\right)\left(\frac{d}{dx} - m_3\right)\ldots\left(\frac{d}{dx} - m_n\right)y = 0 \ldots\ldots(4)$$

Suppose now that $m_{n-1} \equiv m_n \equiv m'$, in which case m' is a root repeated twice over. We have to deal with the equation:

$$a_0\left(\frac{d}{dx} - m_1\right)\left(\frac{d}{dx} - m_2\right)\ldots\left(\frac{d}{dx} - m_{n-2}\right)\left(\frac{d}{dx} - m'\right)^2 y = 0 \ldots(5)$$

Plainly $y = e^{m'x}$ is a solution.

Consider next $y = xe^{m'x}$.

Then
$$\left(\frac{d}{dx} - m'\right)xe^{m'x} \equiv \frac{d}{dx}(xe^{m'x}) - m'xe^{m'x}.$$

$$\therefore \left(\frac{d}{dx} - m'\right)xe^{m'x} = e^{m'x}.$$

Hence, operating on both sides of this last equation by means of $\frac{d}{dx} - m'$, we obtain:

$$\left(\frac{d}{dx} - m'\right)^2 xe^{m'x} = \left(\frac{d}{dx} - m'\right)e^{m'x}.$$

$$\therefore \left(\frac{d}{dx} - m'\right)^2 xe^{m'x} = 0.$$

$$\therefore a_0\left(\frac{d}{dx} - m_1\right)\left(\frac{d}{dx} - m_2\right)\ldots\left(\frac{d}{dx} - m_{n-2}\right)\left(\frac{d}{dx} - m'\right)^2 xe^{m'x} = 0.$$

Thus $xe^{m'x}$ is another solution of (5).

Hence, since $e^{m'x}$ and $xe^{m'x}$ are each solutions of (5), it is plain that the part of the final solution corresponding to the twice-repeated root m' is

$$Ae^{m'x} + Bxe^{m'x}.$$

It is easy to show that $\left(\frac{d}{dx} - m'\right)^3 x^2e^{m'x}$ vanishes identically, and hence that in the case of a thrice-repeated root m' the corresponding solutions are $e^{m'x}$, $xe^{m'x}$, $x^2e^{m'x}$, and therefore that the corresponding part of the final solution is $\mathrm{A}e^{m'x} + \mathrm{B}xe^{m'x} + \mathrm{C}x^2e^{m'x}$, and so on.

Example. *Solve completely the differential equation :*

$$\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 16\frac{dy}{dx} - 12y = 0.$$

Putting $y = e^{mx}$ as a trial solution, we obtain $m = 2, 2, 3$. The corresponding solutions are :

$$e^{2x}, \quad xe^{2x}, \quad e^{3x}.$$

Hence the final solution of the differential equation is :

$$y = \mathrm{A}e^{2x} + \mathrm{B}xe^{2x} + \mathrm{C}e^{3x}.$$

Imaginary Roots for the equation in m.

We require the following results established in last chapter by Taylor and Maclaurin's methods :

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \dots \text{ ad } \infty$$

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \dots \text{ ad } \infty$$

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \dots \text{ ad } \infty.$$

It can be proved that the last result holds good even if x be imaginary. Let us put $x = i\theta$.

$$\therefore \; e^{i\theta} = 1 + i\theta - \frac{\theta^2}{\underline{|2}} - i\frac{\theta^3}{\underline{|3}} + \frac{\theta^4}{\underline{|4}} + \dots \text{ ad } \infty$$

$$= \left(1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} + \dots \text{ ad } \infty\right) + i\left(\theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} + \dots \text{ ad } \infty\right).$$

$$\therefore \; e^{i\theta} = \cos\theta + i\sin\theta.$$

Similarly $$e^{-i\theta} = \cos\theta - i\sin\theta.$$

Hence
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Suppose now that two of the roots of the above equation (3) in m are not real, and that they are $\alpha + i\beta$ and $\alpha - i\beta$, where α, β are both real. The corresponding solutions are:

$$e^{(\alpha+i\beta)x} \text{ and } e^{(\alpha-i\beta)x},$$
$$\textit{i.e. } e^{\alpha x}\times e^{i\beta x} \text{ and } e^{\alpha x}\times e^{-i\beta x}$$
$$\textit{i.e. } e^{\alpha x}(\cos\beta x + i\sin\beta x) \text{ and } e^{\alpha x}(\cos\beta x - i\sin\beta x).$$

Hence their sum (which is free from the imaginary i)

$$\tfrac{1}{2}\{e^{\alpha x}(\cos\beta x + i\sin\beta x) + e^{\alpha x}(\cos\beta x - i\sin\beta x)\} \quad \textit{i.e.} \quad e^{\alpha x}\cos\beta x$$

is also a solution of the given differential equation.

Similarly, their difference (also free from i)

$$\frac{1}{2i}\{e^{\alpha x}(\cos\beta x + i\sin\beta x) \quad e^{\alpha x}(\cos\beta x - i\sin\beta x)\} \quad \textit{i.e.} \quad e^{\alpha x}\sin\beta x$$

is also a solution.

The part of the final solution corresponding to the above pair of conjugate unreal roots is therefore

$$\mathrm{A}e^{\alpha x}\cos\beta x + \mathrm{B}e^{\alpha x}\sin\beta x,$$

an expression not containing the imaginary i.

Example. *Solve the differential equation* $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0.$

Let us assume as a trial solution $y = e^{mx}$. We obtain:

$$m^2 + m + 1 = 0, \quad \textit{i.e.} \quad m = \frac{-1+\sqrt{3}i}{2} \text{ or } \frac{-1-\sqrt{3}i}{2}.$$

The corresponding solutions are:

$$e^{\frac{-1+\sqrt{3}i}{2}x} \text{ and } e^{\frac{-1-\sqrt{3}i}{2}x}$$
$$\textit{i.e. } e^{-\frac{x}{2}}\times e^{+\frac{\sqrt{3}ix}{2}} \text{ and } e^{-\frac{x}{2}}\times e^{-\frac{\sqrt{3}ix}{2}}$$
$$\textit{i.e. } e^{-\frac{x}{2}}\left(\cos\frac{\sqrt{3}x}{2} + i\sin\frac{\sqrt{3}x}{2}\right) \text{ and } e^{\frac{x}{2}}\left(\cos\frac{\sqrt{3}x}{2} - i\sin\frac{\sqrt{3}x}{2}\right).$$

Hence plainly $e^{-\frac{x}{2}}\cos\frac{\sqrt{3}x}{2}$ and $e^{-\frac{x}{2}}\sin\frac{\sqrt{3}x}{2}$ are also each solutions. The final solution is therefore:

$$y=\mathrm{A}e^{-\frac{x}{2}}\cos\frac{\sqrt{3}x}{2}+\mathrm{B}e^{-\frac{x}{2}}\sin\frac{\sqrt{3}x}{2},\ \text{(free from } i\text{)}.$$

We shall now show how to solve differential equations of the type:

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\dots+a_ny=f(x).$$

We first of all obtain *by trial or inspection* a solution $y=\phi(x)$ as simple as possible.

We then take $y=z+\phi(x)$ and substitute in the given equation. We thus obtain a differential equation in z in which the right-hand side is zero, since $\phi(x)$ is known to be a solution of the given differential equation, and proceed as explained above. A few examples will make the method of procedure plain.

Example 1. *Solve* $\frac{d^2y}{dx^2}-2\frac{dy}{dx}-8y=x^2.$

Trial Solution.—Since an expression in x of the fourth degree when differentiated twice gives an expression in x of degree two, let us assume as a *trial solution* that y is of the form $ax^4+bx^3+cx^2+dx+e$, and substitute in the given equation. Substituting in the given equation and collecting coefficients of the various powers of x, we obtain:

$$-8ax^4-(8a+8b)x^3+(12a-6b-8c)x^2+(6b-4c-8d)x \\ +(2c-2d-8e)\equiv x^2.$$

Hence, equating the coefficients of the various powers of x on both sides, we get the following series of equations in a, b, c, d, e:

$$-8a=0$$
$$-8a-8b=0$$
$$12a-6b-8c=1$$
$$6b-4c-8d=0$$
$$2c-2d-8e=0.$$

Hence $\quad a=0,\quad b=0,\quad c=-\frac{1}{8},\quad d=\frac{1}{16},\quad e=-\frac{3}{64}.$

A particular solution is therefore:

$$-\frac{1}{64}(8x^2-4x+3).$$

Let us take, therefore, $y=z-\frac{1}{64}(8x^2-4x+3)$ and substitute in the given equation. We obtain an equation in z whose left-hand side is zero, since $-\frac{1}{64}(8x^2-4x+3)$ is known to be a solution of the given equation. Hence:

$$\frac{d^2z}{dx^2}-\frac{2dz}{dz}-8z=0.$$

Solving as before explained we get:

$$z=\text{A}e^{4x}+\text{B}e^{-2x}.$$

$$\therefore\ y=\text{A}e^{4x}+\text{B}e^{-2x}-\frac{1}{64}(8x^2-4x+3)$$

is the complete solution of the given differential equation.

Example 2. *Solve* $\frac{d^2y}{dt^2}-2\frac{dy}{dt}+5y=\sin 3t.$

Trial Solution.—We know that $\sin 3t$ or $\cos 3t$ when differentiated any number of times give $\sin 3t$ or $\cos 3t$. We therefore assume as a trial solution $a\sin 3t+b\cos 3t$ and insert in the given differential equation, obtaining:

$$-9(a\sin 3t+b\cos 3t)-6(a\cos 3t-b\sin 3t)+5(a\sin 3t+b\cos 3t)\equiv\sin 3t,$$

$$i.e.\ \ (-4a+6b)\sin 3t+(-4b-6a)\cos 3t\equiv\sin 3t.$$

Equating coefficients of $\sin 3t$ and $\cos 3t$ on both sides, we obtain:

$$-4a+6b=1$$

$$-6a-4b=0,$$

$$i.e.\ \ a=-\frac{1}{13},\qquad b=\frac{3}{26}.$$

$\therefore\ -\frac{1}{13}\sin 3t+\frac{3}{26}\cos 3t$ is a particular solution.

Hence, put $$y=z-\frac{1}{13}\sin 3t+\frac{3}{26}\cos 3t.$$

$$\therefore\ \frac{d^2z}{d^2t}-2\frac{dz}{dt}+5z=0.$$

Putting $z = e^{mt}$, we have $m^2 - 2m + 5 = 0$

or
$$(m-1)^2 + 4 = 0.$$

$$\therefore\ m = 1 \pm 2i.$$

$$\therefore\ z = e^t(\mathrm{A}\cos 2t + \mathrm{B}\sin 2t),\ \text{(free from } i\text{)}.$$

Hence finally $y = e^t(\mathrm{A}\cos 2t + \mathrm{B}\sin 2t) - \frac{1}{13}\sin 3t + \frac{3}{26}\cos 3t$

is the general solution, free from the imaginary i.

EXAMPLES XX.

1. Find the general equation to the family of curves defined by the differential equation $\frac{dy}{dx} = 3$, and interpret these curves geometrically by drawing a few of them.

2. Find the general equation to the family of curves defined by the differential equation $y\frac{dy}{dx} + x = 0$, and interpret these curves geometrically by drawing a few of them.

3. Obtain the general equation to the family of curves $\frac{dy}{dx} + \frac{3}{y} = 0$. What is the geometrical interpretation of this family?

4. What family of curves is defined by the differential equation $x\frac{dy}{dx} + y = 0$? Sketch a few members of the family on squared paper.

5. Prove that *two* members of the family of curves represented by the differential equation $x\left(\frac{dy}{dx}\right)^2 - 3\left(\frac{dy}{dx}\right) + 2y = 0$ pass through every point of the plane of the axes.

Find the angle at which the two members passing through the point (1, 1) cut one another.

6. Prove that two members of the family of curves represented by the differential equation $(x^2-1)\left(\frac{dy}{dx}\right)^2 - 3x\frac{dy}{dx} + y^2 = 0$ pass through every point of the plane of the axes.

Find the locus of the points, the curves through which cut at right angles.

7. Find the locus of points at which the tangents to the family of curves $(3x - 4y + 5)\frac{dy}{dx} = (x - 2y + 3)$ are parallel to the axis of x.

8. In the preceding question, find the locus of points at which the tangents are parallel to the axis of y.

9. Solve the equations: (i) $\frac{dy}{dx}+\frac{1+y^2}{1+x^2}=0$, (ii) $\frac{dy}{dx}+\frac{\tan y}{\tan x}=0$,

(iii) $\frac{dy}{dx}=\frac{(1+x)^7}{(1-y)^3}$, (iv) $\frac{dy}{dx}=\frac{(1-y)^3}{(1+x)^7}$, (v) $\frac{dy}{dx}=\frac{(1-y^2)^{\frac{1}{2}}}{e^x}$.

10. Solve the equations: (i) $\frac{dy}{dx}+x^2y=x^2$, (ii) $\frac{dy}{dx}+xy=x^3$,

(iii) $\frac{dy}{dx}+\frac{y}{x^2}=\frac{1}{x^2}$, (iv) $\frac{dy}{dx}+y\cos x=\sin x\cos x$,

(v) $\frac{dy}{dx}-y\sin x=-\sin x\cos x$.

11. Solve the differential equation

$$(x+1)(x+2)\frac{dy}{dx}+2y=(x+2)(x+3). \qquad \text{(Math. Trip.)}$$

12. Solve the linear equation $\frac{dy}{dx}+2y\tan x=\sin x$, and show that, if $y=0$ when $x=\frac{1}{3}\pi$, the maximum value of y is $\frac{1}{8}$. (Math. Trip.)

13. Given $\frac{dx}{dt}+\mu x=t$, find x in terms of t, given that $x=0$ when $t=0$. Find also the value of x when $t=\infty$.

14. Given that $\frac{dy}{dx}+\alpha y=x^2$, α being a constant, and that $y=0$ when $x=0$, find y in terms of x, and also the value of y when $x=\infty$.

15. An electric current of C amperes is being driven through a coil of resistance R ohms and self-inductance L henrys by a cell whose electromotive force is E volts. The differential equation giving the current before it becomes steady is $\mathrm{L}\frac{d\mathrm{C}}{dt}+\mathrm{RC}=\mathrm{E}$. Find the current after t seconds, and its ultimate value when t is very large (*i.e.* $t=\infty$).

16. Solve the differential equation $\frac{dy}{dx}+\frac{y}{x}=4x^2$.

17. Integrate completely the differential equation $\frac{dy}{dx}-y\tan x=1$.

18. Integrate $\cos x\frac{dy}{dx}+y\sin x=1$.

19. Integrate $\sin x \dfrac{dy}{dx} - y \cos x + 1 = 0$.

20. Integrate $\dfrac{dy}{dx} - y = e^x \cos x$.

21. Integrate $\tan x \dfrac{dy}{dx} - y = \sin^2 x$.

22. Find the family of curves represented by $\dfrac{dy}{dx} = \dfrac{3x - 2y}{x}$.

23. Integrate the differential equation $\dfrac{dy}{dx} = \dfrac{y}{x - y}$.

24. Integrate the differential equation $\dfrac{dy}{dx} = \dfrac{x + y}{x - y}$.

25. Solve the equations: (i) $\dfrac{dy}{dx} = \dfrac{2y - x}{y}$, (ii) $\dfrac{dy}{dx} = \dfrac{x^2 + 2y^2}{xy}$.

26. Integrate $y = px - 5p^3$. What is the singular solution? $\left[N.B.\text{—}p \equiv \dfrac{dy}{dx}.\right]$

27. Integrate $y = px + (1 + p^2)$. What is the singular solution?

28. Integrate $y = px + p + p^2$. What is the singular solution?

29. Integrate $y = px + 3p^2$. What is the singular solution?

30. Find the general solution of the differential equation $y = px - \dfrac{4}{p}$. What is the singular solution?

31. Solve the equations:

$$\text{(i) } y = x\frac{dy}{dx} + \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}, \quad \text{(ii) } y = x\frac{dy}{dx} + \frac{1}{\dfrac{dy}{dx}},$$

$$\text{(iii) } y = x\frac{dy}{dx} + \left\{\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2\right\}, \quad \text{(iv) } y = x\frac{dy}{dx} - \left\{2\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3\right\},$$

$$\text{(v) } y = x\frac{dy}{dx} + \frac{\dfrac{dy}{dx}}{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}},$$

giving both general and singular solution.

32. Solve graphically the differential equation $(x+y)^2 \frac{dy}{dx}=25$, and obtain approximately the curve of the system that passes through the point (3, 4).

33. Prove that, if $y^3-3ax^2+x^3=0$,

$$\frac{d^2y}{dx^2}+\frac{2a^2x^2}{y^5}=0.$$

Show that the curve given by the above equation is everywhere concave to the axis of x, and that there is a point of inflexion where $x=3a$. (Math. Trip.)

34. Prove that $\frac{1}{8}(35\cos^4 x-30\cos^2 x+3)$ ranges in value between unity and $-\frac{3}{7}$, and has also $\frac{3}{8}$ as a maximum value. (Math. Trip.)

35. A certain curve passes through the origin and is such that the rectangle contained by a fixed line a and the subnormal is always equal to the square on the abscissa. Find the equation to this curve.

36. There is only one type of curve whose subnormal is proportional to its subtangent. Find this type.

37. There is only one type of curve whose subnormal is proportional to its ordinate. Find this type of curve.

38. Find the species of curve which possesses the property that its subnormal is proportional to its abscissa.

39. A curve passes through the origin and possesses the property that the sum of its subnormal and subtangent is always equal to 2. Find the equation to the curve.

40. Find the curve which passes through the point (1, 1) and is such that the subnormal is always equal to the abscissa.

41. A certain curve touches the axis of x at the origin and is such that the radius of curvature at any point is equal to $(1+x^2)^{\frac{3}{2}}$. Find the equation to the curve.

42. Find the family of curves whose subtangent is constant.

43. Given that $\rho \cos x=1$, x in radians, and that the curve touches the axis of x at the origin, find the equation to the curve.

44. Solve completely $\frac{d^2y}{dx^2}=\frac{dy}{dx}$.

45. Integrate completely $r\frac{d^2V}{dr^2}+2\frac{dV}{dr}=0$. $\left[\textit{Hint.}\text{—Let } \frac{dV}{dr}=U.\right]$

46. Obtain a first integral $\left(i.e. \text{ an integral involving } \frac{du}{d\theta}\right)$ of $\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2}$, where μ and h are each constants.

[*N.B.*—This equation occurs in discussing the motion of the planets.]

47. Solve $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0$.

48. A particle starting from rest a very long way off moves towards the earth under the influence of gravity with a motion defined by the differential equation $\frac{d^2r}{dt^2}=-g\frac{a^2}{r^2}$, where r is the distance of the particle from the centre of the earth after time t, a is the radius of the earth, and g the value of gravity at the surface of the earth. Find the speed with which the particle would reach the surface of the earth, and also its distance from the centre of the earth after time t.

49. A particle moves so as to have its motion retarded in such a way that the retardation is proportional to its velocity. The differential equation defining the motion is $\frac{d^2x}{dt^2}+\mu\frac{dx}{dt}=0$. If the particle has a velocity v_0 at the start of its motion and starts from the position $x=a$, find its position x at any time t. Also find v in terms of x. What will be the exact state of affairs after a very long time ($i.e.$ $t=\infty$)?

50. (i) Obtain the third Taylor approximating-curve to the curve passing through the point $(0, +1)$ as defined by the differential equation $\frac{dy}{dx}=y+x^2+y^2$.

(ii) Hence calculate the value of y when x has the value 0·03, giving your answer correct to four places of decimals.

51. Find the fourth Taylor approximating-curve at the point $\left(\frac{\pi}{2}, 1\right)$ to the curve of the system $x\frac{dy}{dx}=x^2\cos x+y$.

52. (i) Find the third Taylor approximating-curve to that curve of the system $(3x+4)\frac{dy}{dx}=(4x-2y+5)$ which passes through the origin.

(ii) Hence calculate the value of y when $x=0{\cdot}04$, giving your answer correct to five places of decimals.

(iii) Find also the third Taylor approximating-curve to that curve of the above system which passes through the point $(1, 1)$.

53. In the theory of the simple pendulum occurs the differential equation $\frac{d^2\theta}{dt^2}+\frac{g}{a}\sin\theta=0$. Given that $\frac{d\theta}{dt}$ vanishes when $\theta=\alpha$, find an expression giving $\frac{d\theta}{dt}$ in terms of θ and α.

[*N.B.*—The equation obtained cannot be further integrated unless one knows Elliptic Functions.]

54. A particle moves so as to have its motion retarded in such a way that the resistance varies as the square of the velocity. The differential equation defining this motion is known to be $\frac{d^2x}{dt^2}=-\mu\left(\frac{dx}{dt}\right)^2$. The particle starts from a distance a with a velocity V. Find v in terms of x, and x in terms of t.

55. A particle moves in a straight line under a retardation $\mathrm{K}v^{m+1}$, where v is the velocity at time t. Show that, if u is the velocity at $t=0$,

$$\mathrm{K}t=\frac{1}{m}\left(\frac{1}{v^m}-\frac{1}{u^m}\right),$$

and obtain a corresponding formula for the space described in terms of v and u.

A bullet fired with a horizontal velocity of 2500 ft. per sec. is travelling with a velocity of 1600 ft. per sec. at the end of one second. Assuming that $m=\frac{1}{2}$, calculate K and the space traversed in the first second, neglecting the effect of gravity. (Math. Trip.)

56. The following equation occurs in studying the motion of a smooth ring on a vertical circular hoop under the action of an elastic string:

$$a\ddot{\theta}+g\sin\theta=\frac{\lambda}{ml}\sin\frac{\theta}{2}\left(2a\cos\frac{\theta}{2}-l\right).$$

Express $\dot{\theta}$ in terms of θ, given that $\dot{\theta}$ vanishes when $\theta=\alpha$.

57. Find the linear differential equation satisfied by $\frac{\mathrm{A}}{x}\cos(\mu x+\alpha)$ for all values of A and α. (Math. Trip.)

58. A curve is determined by the property that the tangent to the curve at any point P meets a fixed straight line in T, so that the length PT is constant. Show that the radius of curvature of the curve at P is $\frac{\mathrm{PT}\,.\,\mathrm{TN}}{\mathrm{PN}}$, where PN is the perpendicular from P to the fixed straight line. (Math. Trip.)

59. The following equation occurs in studying the motion of a ball projected vertically upwards with a chain of indefinite length attached to it, x being the height of the ball above the ground :

$$\frac{d}{dt}\left\{(c+x)\frac{dx}{dt}\right\}=-(c+x)g$$

Obtain a first integral of the motion.

60. The equation $\frac{d^2x}{dt^2}+c^2(x-a)=0$ represents the motion of a particle tied to an elastic string on a smooth table. Find the complete integral.

61. Integrate completely $\left.\begin{aligned}\frac{d^2x}{dt^2}&=0\\ \frac{d^2y}{dt^2}&=-\frac{1}{y^3}\end{aligned}\right\}.$

62. The following equation occurs in connection with the rotation of a glass tube in a vertical plane, with a bullet moving inside it :

$$\frac{d^2r}{dt^2}-\omega^2r=-g\sin\omega t.$$

Integrate completely.

63. Integrate completely $\frac{dx}{dt}=y$

$$\frac{dy}{dt}=x.$$

64. Integrate completely $4\frac{d^2z}{dt^2}-12\frac{dz}{dt}+9z=0.$

65. Integrate $\frac{d^2y}{dx^2}+5\frac{dy}{dx}-6y=0.$

66. Integrate $8\frac{d^2y}{dx^2}-14\frac{dy}{dx}+5y=0.$

67. Solve $\frac{d^2y}{dx^2}-\frac{dy}{dx}-6y=2x.$

68. Solve $\frac{d^2y}{dx^2}-7\frac{dy}{dx}+12y=3x+4.$

69. Solve $\frac{d^2y}{dx^2}-2\frac{dy}{dx}-15y=3x^2.$

70. Solve $\frac{d^2y}{dx^2}-4\frac{dy}{dx}=3\sin 2x.$

71. Solve $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = \sin t$.

72. Solve $\frac{d^2y}{dt^2} + 4y = e^{3t}$. [*Hint.*—Put $y = ze^{3t}$.]

73. Solve $\frac{d^2y}{dt^2} - 9y = 2e^t - 3e^{2t}$.

74. Solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = te^t$. [*Hint.*—Put $y = ze^t$.]

MISCELLANEOUS EXAMPLES III.

CHAPS. XI.–XV.

106. If $y=x\sin x$, prove that $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+y(x^2+2)=0$.

107. The portion of the curve $y=5\sin 2x$ between $x=0$ and $x=\frac{\pi}{4}$ revolves about the axis of x. Find the volume of the solid formed, and determine the position of the centre of gravity of the solid. (Q.P.M.S.T.)

108. Determine the function $\frac{ax^2+2bx+c}{a'x^2+2b'x+c'}$ which has turning values 2 and 3 when $x=1$ and $x=-1$ respectively, and has the value 2·5 when $x=0$. (M.T.)

109. Prove that the minimum value of $\frac{1-x+x^2}{1+x-x^2}$ is $\frac{3}{5}$.

110. A circular disc can turn about a smooth pivot through its centre on a rough horizontal table. The pressure of the disc on the table is distributed uniformly. Show that, if μ be the coefficient of friction and W the weight of the disc, the least force that will turn the disc round the pivot is $\frac{2}{3}\mu$W. (Queens'.)

111. If $y=\sqrt{\frac{1-x}{1+x}}$, prove that $(1-x^2)\frac{dy}{dx}+y=0$.

112. Evaluate $\int_0^{\frac{\pi}{2}}(\sin px+\cos px)\,dx$.

113. Find the area of the loop of the curve $y^2=\frac{x^2(1+x)}{1-x}$. (Q.P.M.S.T.)

114. If $x=3\sin 2t-\cos 2t$, where t represents seconds, what is the least value of the time for which $\frac{dx}{dt}=0$, and what is the value of $\frac{d^2x}{dt^2}$ at that time? (Q.P.M.S.T.)

115. If $y=x\tan^{-1}x$, prove that $(1+x^2)^2\dfrac{d^2y}{dx^2}=2.$

116. A right circular cone is to be constructed with a given curved surface, so that the sphere inscribed in the cone and touching its base may be as large as possible. Show that the vertical angle of the cone must be $2\sin^{-1}(\sqrt{2}-1)$. (St. Catherine's.)

117. Sketch the curve $\left\{\begin{array}{l} x=a\sin 2t \\ y=a\sin t \end{array}\right\}$ and prove that the area of a loop of this curve is $\frac{4}{3}a^2$.

118. In a range-finder the base line AB is 10 yards long. The line AP, to the point whose range is wanted, is at right angles to AB, while the angle PBA is 88·2°.

If the measurement of the angle PBA was wrong by the tenth of a degree, what would the error in the range be? (Army.)

119. Find the differential coefficients of

$$\text{(i)}\quad \frac{x-2}{\sqrt{2x^2-2x+5}}$$

$$\text{(ii)}\quad \sin\frac{1+x^2}{1+x^3}.$$

120. An angle α is divided into two parts so that the ratio of the sines of the parts has a given value K. Show that as α changes from 0 to π, and K is fixed and less than unity, the less part has a maximum $\sin^{-1}$ K. (Pembroke.)

121. Prove that, as x increases, $\dfrac{a\sin x+b\cos x}{c\sin x+e\cos x}$ either increases for all values of x or else decreases for all values of x; and find the condition to be satisfied by the constants a, b, c, e to provide that it shall always increase. (Q.P.M.S.T.)

122. A, B, and C are three stations on a level piece of ground. AB has been measured accurately, and its length is known to be 1251 links. The position of C has been determined by measuring the angles BAC and ABC. The angle BAC has been measured as 45° 40′, and this is taken to be accurate. The angle ABC has been measured as 60° 30′, but it is supposed that there may be an error of 3 minutes in the measurement. Express the possible error in AC as a percentage of the length AC. (Army Adapted.)

123. If $y=\dfrac{ax}{x+b}$, prove that $\dfrac{2}{y}\dfrac{dy}{dx}-\dfrac{\dfrac{d^2y}{dx^2}}{\dfrac{dy}{dx}}=\dfrac{2}{x}$. (Q.P.M.S.T.)

124. Differentiate $e^{-x}(\mathrm{A}\cos x+\mathrm{B}\sin x)$ where A and B are constants Choosing suitable values for A and B, hence find the value of $\int_0^{\frac{\pi}{2}} e^{-x}\cos x\,dx$. (M.T.)

125. Sketch the curve $y^2=x^2(1-x^2)$, and show that the area of either loop is $\frac{2}{3}$. (Oxf. and Camb. Joint Board.)

126. Evaluate $\int_0^1 x^2\sin^{-1}x\,dx$. (Oxf. and Camb. Joint Board.)

127. Trace the Lemniscate of Bernoulli given by the polar equation $r^2=\cos 2\theta$, and find the area enclosed by a loop of the curve. (Army Adapted.)

128. Sketch the shape of the curve $y=a\cos^2\frac{x}{b}$, and show that the area of each of the regions bounded by the curve and the x-axis is $\frac{1}{2}\pi ab$. (Q.P.M.S.T.)

129. Show that $\frac{d}{dx}\tan^{-1}(\frac{1}{2}\tan x)=\frac{2}{4-3\sin^2 x}$. (Oxf. and Camb. Joint Board.)

130. A ladder is to be carried in a horizontal position round a corner formed by two streets a ft. and b ft. wide meeting at right angles. Prove that the length of the longest ladder that will pass round the corner without jamming is $(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}$ ft. (Oxf. and Camb. Joint Board.)

131. Evaluate the integrals :

$$\text{(i)}\ \int x\sin^2 x\,dx$$

$$\text{(ii)}\ \int_0^1\frac{dx}{1+x+x^2}$$

$$\text{(iii)}\ \int_0^{\frac{\pi}{4}}\sin x\sin 2x\,dx.$$

(Q.P.M.S.T.)

132. Find the maximum and minimum values of $\frac{x^2+x+1}{x^2-1}$, and the corresponding values of x. Illustrate by a figure. (Trinity.)

133. Prove that $\int_1^2\frac{dx}{\sqrt{(2-x)(x-1)}}=\pi$.

134. If p, the pressure, and v, the volume of a gas are connected by the relation $pv^{1\cdot 4}=$constant, find the percentage increase in the pressures corresponding to a diminution of $\frac{1}{2}\%$ in the volume. (Oxf. and Camb. Joint Board.)

135. Prove that $\int_1^2 \frac{1}{x^2+2x+3}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{4\sqrt{2}}\right)$.
(Oxf. and Camb. Joint Board.)

136. Evaluate $\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^2\theta\,d\theta$. (Selwyn.)

137. An arc of a circle of radius a is of length $a\alpha$. Find the volume generated by revolving the arc about its chord. (Magdalene.)

138. Prove $\int_{\frac{1}{2}}^1 \sqrt{1-x^2}\,dx = 0{\cdot}307\ldots$ (Trinity.)

139. The deflection at the centre of an iron rod of length l and diameter d, supported at its ends and loaded at the centre with a weight W, is known to be proportional to $\frac{Wl^3}{d^4}$. What is the percentage increase in the deflection if the load W is increased 3%, the length l is increased $1\frac{1}{2}$%, and the diameter increased 1%?
(Q.P.M.S.T.)

140. Find the volumes of the following solids:

(i) The shuttle-shaped solid generated by the revolution about OX of the curve $y=\frac{\sin x}{3}$ between $x=0$ and $x=\pi$.

(ii) The lens-shaped solid generated by the revolution about OY of the curve $y=\frac{\cos x}{3}$ between $x=0$ and $x=\frac{\pi}{2}$. (Q.P.M.S.T.)

141. The x, y coordinates of a point are given by the expressions

$$x = 1-\cos\theta$$
$$y=\cos\theta+\tfrac{1}{4}\cos 2\theta.$$

Prove that $\frac{dy}{dx}$ is equal to $-2\cos^2\frac{1}{2}\theta$; and sketch the path described by the point as θ increases from 0° to 180°. What is the path of the point as θ increases from 180° to 360°? (Q.P.M.S.T.)

142. Show that the expression $\tan^{-1}x-\frac{x}{1+x^2}$ is always positive as x changes from 0 to $\frac{\pi}{2}$. (Q.P.M.S.T.)

143. Prove that the equation of the tangent at the point $(4am^2, 8am^3)$ of the curve $ay^2=x^3$ is $y=3mx-4am^3$. (Downing.)

144. The base a and base angles θ, ϕ of a triangle are given. Find the error in the altitude of the triangle due to small errors in the measurements of θ and ϕ. (Pembroke.)

145. Evaluate $\int x^3 \cos x \, dx$. (Oxf. and Camb. Joint Board.)

146. Find the volume of the solid formed by a complete revolution about the axis of x, of the portion of the curve $y=a \sin \frac{\pi x}{b}$ between $x=0$ and $x=b$. (Q.P.M.S.T.)

147. Prove that $\int_0^\pi \frac{1}{5+3\cos x} \, dx=\frac{\pi}{4}$.

(Oxf. and Camb. Joint Board.)

148. Show that $\int_0^{\frac{\pi}{2}} \frac{x \cos \theta \, d\theta}{1-x^2\cos^2\theta}=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$.

149. Find the equation of the tangent at the point $\left(\frac{a}{m}, 2am^2\right)$ on the curve $x^2y=2a^3$, and prove that it meets the curve again in the point $\left(-\frac{a}{2m}, 8am^2\right)$.

150. In a triangle one side is measured as 16·3 inches, another side as 10·5 inches, and the angle between them as 56° 30′. Calculate the area of the triangle.

If the correct lengths of the sides are 16·5 inches and 10·4 inches, find the percentage error in the calculated area.

Also find the percentage error in the calculated area arising from an error of 2 minutes in the measurement of the angle, the sides being taken as correctly measured. (Q.P.M.S.T.)

151. Prove $\int_{\frac{1}{2}}^1 \frac{\sqrt{1-x^2}}{x^2} dx=0{\cdot}685\ldots$ (Trinity.)

152. Find the area A between the curve $y=a(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x)$ and the x-axis between the limits 0 and π, and the volume V obtained by rotating this area about the x-axis. Prove that $4\text{V}=\pi^2 a\text{A}$.

(Math. Trip.)

153. Prove that $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$.

Apply this theorem to prove that $\int_0^{\frac{\pi}{2}} \frac{2\sin x+3\cos x}{\sin x+\cos x} dx=\frac{5\pi}{4}$.

(Oxf. and Camb. Joint Board.)

154. (i) The angle C and the sides a, b of a triangle are measured and found to be 90°, 2000 ft. and 3000 ft. respectively. Find the greatest possible error in the estimate of area based on these figures, having given that the measurements of the sides may each be in error by $\frac{1}{10}$ ft.

(ii) It is found that the actual error in the estimate of area is about twice the greatest amount derivable from these errors in the measurements of the sides; what is the least error in the measurement of the angle that will account for this? (Q.P.M.S.T.)

155. Evaluate $\int_0^{\frac{\pi}{2}} \sin mx \,.\, \sin nx\, dx.$ (Math. Trip.)

156. Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 mx\, dx.$ (Math. Trip.)

157. Find the area included between the axis of x and one semi-undulation of the curve $y=c\sin(x-a)/b$.
(Oxf. and Camb. Joint Board.)

158. Show that $\int_0^3 \frac{\sqrt{9-x^2}}{25-x^2}dx=\frac{\pi}{10}$.
(Oxf. and Camb. Joint Board.)

MISCELLANEOUS EXAMPLES IV.

CHAPS. XVI.-XX.

159. Given that $(ds)^2=(dx)^2+(dy)^2$ and $\rho=\frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\ddot{x}\dot{y}}$, show that for the cycloid $x=a(t+\sin t)$, $y=a(1-\cos t)$, the length of the arc, measured from the point for which $t=0$, is $4a\sin\frac{1}{2}t$, and the radius of curvature is $4a\cos\frac{1}{2}t$. $\left(N.B.\text{—}\dot{x}=\frac{dx}{dt}, \text{ etc.}\right)$ (M.T.)

160. If x is so small that its cube and higher powers may be neglected, show that

$$\frac{(9+5x)^{\frac{1}{2}}(1-x^2)^{\frac{3}{2}}}{(8-x)^{\frac{1}{3}}}=\frac{3}{16}\left[1+\frac{4}{9}x-\frac{3805}{2592}x^2\right]. \quad \text{(Q.P.M.S.T.)}$$

161. In the surface formed by the revolution of $y^2=x^3$ about the x-axis, what is the rate of increase of the surface cut off by a plane perpendicular to the axis of x when $x=2$?

162. Discuss by means of the differential coefficient the variation in value of $x\log_e x$ for positive values of x, and show that $x\log_e x=c$ has one real root if c is positive, and two real roots if c is negative and numerically less than e^{-1}. (Pembroke.)

163. Evaluate (i) $\int x \sin^2 x \, dx$

(ii) $\int \frac{x}{\sqrt{1-x}} \, dx$

(iii) $\int_{-1}^{+1} \frac{x^2 \, dx}{\sqrt{1-x^2}}$ (King's.)

164. Trace the curve $3ay^2 = x(a-x)^2$, and find the length and area of its loop. (Oxf. and Camb. Joint Board.)

165. Evaluate $\int_0^1 \frac{(x+1)}{(x+2)(x+3)} \, dx$.

166. Plot the curve

$$y^2(1+x^2) = (3-x)(x-2),$$

and show that the volume generated by revolving this curve about the axis of x is

$$\frac{\pi}{2}[5 \log_e 2 - 2 - 10 \tan^{-1} \tfrac{1}{7}].$$ (Q.P.M.S.T.)

167. Show that approximately, when x is small,

$$x \cot x = 1 - \frac{x^2}{3} - \frac{x^4}{45},$$

higher powers of x than x^5 being neglected. Find cot 5° to four decimal places. (Oxf. and Camb. Joint Board.)

168. Show that the distance from the origin of the centre of gravity of the area included between the curve $(x^2+c^2)y = c^3$ and the whole length of the axis of x is $\frac{c}{4}$. (Oxf. and Camb. Joint Board.)

169. Find the indefinite integrals of $\sec x$, $\frac{1}{1+\cos x}$, and show that

$\int_0^1 \frac{dx}{\sqrt{1+x}+\sqrt{x}} = \frac{4}{3}(\sqrt{2}-1)$. (Oxf. and Camb. Joint Board.)

170. Show that the length of one quadrant of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is equal to $\frac{3a}{2}$, and find the length of one quadrant of the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$$ (M.T.)

171. State Taylor's theorem, and apply it to express $\sin(\frac{1}{4}\pi + \theta)$ in a series of powers of θ. (Math. Trip.)

172. Differentiate (i) $\log_e(\tan x + \sec x)$,

$$\text{(ii)}\ \frac{1+3x}{(1+4x+6x^2)^{\frac{1}{2}}}.$$

Find the slope of the curve $y = e^{-\frac{x}{10}} \sin 5x$ at the point $(\pi, 0)$.
(Q.P.M.S.T.)

173. Evaluate $\int \frac{x\,dx}{(x+1)^2(x-1)}$.

174. Trace the curve given by the equation $r = a(1+\cos\theta)$. Find the length of the curve and the area enclosed by the curve.
(Oxf. and Camb. Joint Board.)

175. Express $\frac{2x+3}{x^3+x^2-2x}$ as the sum of its partial fractions, and find the integral $\int \frac{2x+3}{x^3+x^2-2x}dx$. (Army.)

176. Evaluate

$$\text{(i)}\ \int_5^6 \frac{x^2\,dx}{x-2}, \quad \text{(ii)}\ \int_0^{\frac{\pi}{3}} \sin x \cos 2x\,dx, \quad \text{(iii)}\ \int x \log_e x\,dx.$$

177. Plot the graph of the equation $y = 100\log_{10} x - x$ from $x = 10$ to $x = 150$.

From the graph determine approximately the gradient of the curve at the points for which $x = 20$ and $x = 60$. By aid of the calculus calculate these values more accurately, and also find the value of x which makes y a maximum. (Army Adapted.)

178. Prove the formula for the determination of the length of an arc of a curve whose equation in polar coordinates is given.

Find the length of the arc of the curve

$$r\cos^3\theta = a\sin^2\theta$$

cut off by the straight lines $\theta = 0$, $3\tan\theta = \sqrt{5}$.
(Oxf. and Camb. Joint Board.)

179. Use your exponential tables to trace roughly the curve $r = 3e^\theta$ on squared paper.

(i) Calculate the angle between the radius vector and the tangent at any point P on the curve, and hence show that the value of this angle is constant.

(ii) Prove that the area enclosed between the curve and the radii vectores OP_1 and OP_2 is $\frac{1}{4}({r_2}^2 - {r_1}^2)$.

(iii) Prove that the arc P_1P_2 is equal in length to $\sqrt{2}$ times the difference between the radii vectores drawn to P_1 and P_2 respectively.

180. Expand by Maclaurin's method $\tan^2 x$ in ascending powers of x as far as x^3.

181. The equation of a rectangular hyperbola is $xy=6$. Calculate the area included between the curve and the part of the x-axis from $x=1$ to $x=6$. You may take $e=2{\cdot}718$. (Army.)

182. Evaluate $\int_0^x \frac{2x+1}{x+2}dx$ (sketching the graph first of all).

183. Show that as x increases from the value -1 the function $(1+x)^{-\frac{1}{x}}$ assumes once and once only every value lying between 0 and 1. (M.T.)

184. Evaluate the integrals:

$$\text{(i)} \int (x^2-a^2)^{-\frac{3}{2}}\,dx$$

$$\text{(ii)} \int (x^3-1)^{-1}\,dx.$$

185. (i) Find two independent solutions of

$$\frac{d^2x}{dt^2}+3\frac{dx}{dt}-10x=0$$

by substituting $x=e^{mt}$.

(ii) Hence find the solution of the above equation for which $x=a$ and $\frac{dx}{dt}=v$ when $t=0$. (Adapted, M.T.)

186. Find the values of $\frac{dy}{dx}$ and of $\frac{d^2y}{dx^2}$ when

$$\text{(i)}\ y=e^{2x}\sin 3x$$

$$\text{(ii)}\ y=\sqrt{1+x+x^2}.$$

187. Trace roughly the curve $\cot x-\frac{1}{x}$ on squared paper, and hence show graphically that there exists a tangent to the curve where it cuts the y-axis.

Hence find Maclaurin's third approximating-curve at the point where the curve cuts the y-axis (*i.e.* expand as far as x^3).

188. Show that the surface generated by the revolution of the cardioid $r=a(1-\cos\theta)$ about the line $\theta=0$ is $\frac{32}{5}\pi a^2$. (Trinity).

189. Find the first and second differential coefficients of

(i) $(1-x^2)\log_e(1+x)$

(ii) $\dfrac{1-\sin x}{1+\sin x}$ (Q.P.M.S.T.)

190. Find by Maclaurin's theorem the expansion of $\dfrac{e^x}{e^x+1}$ as far as the term in x^3; and show that no even powers of x can occur in the expansion to any number of terms. (Math. Trip.)

191. Evaluate the integrals:

(i) $\displaystyle\int_0^1 x\log_e(1+x)\,dx$

(ii) $\displaystyle\int\frac{dx}{(x+a)(x+b)}$. (Q.P.M.S.T.)

192. Find *two* independent solutions of the equation

$$\frac{d^2x}{dt^2}-2\frac{dx}{dt}-3x=0$$

by substituting $x=e^{mt}$.

193. Prove that $\log_e x+\dfrac{1}{x}$ is not less than 1 for any positive value of x. (Q.P.M.S.T.)

194. Determine the stationary values of the function $y=e^{-ax}\sin bx$, where a and b are positive.

Show that these values form a geometrical progression whose common ratio is $-e^{-\frac{\pi a}{b}}$; and illustrate your results by a figure. (M.T.)

195. Show that the function

$$y=ax^2+2bx+c+\mathrm{A}\cos mx+\mathrm{B}\sin mx$$

satisfies an equation of the form

$$\frac{d^4y}{dx^4}+k^2\frac{d^2y}{dx^2}=l$$

and determine the values of k and l. (Trinity.)

196. Integrate with respect to x:

(i) $\dfrac{x^3+1}{x+2}$

(ii) $\dfrac{(a^x+b^x)^2}{a^xb^x}$

(iii) $\cos^2 2x\sin 3x$. (Peterhouse.)

197. Calculate the maximum and minimum values (if any) of $x^2 \log_{10} x$ to three places of decimals. Show that each negative value of this function corresponds to two distinct values of x, and each positive value of it to only one value of x. For what value of a does the curve $y=\frac{a}{x^2}$ touch the curve $y=\log_{10} x$? (Q.P.M.S.T.)

198. Obtain five terms of the infinite series by which $(1+x)^{\frac{1}{2}}$ can be expanded in ascending powers of x when x is small. Apply it to calculate $(1{\cdot}15)^{\frac{1}{2}}$ to three decimal places. (Army.)

199. Using the method of "integration by parts," evaluate $\int e^{2x} \cos 3x\, dx$ and $\int x^3 \sin x\, dx$.

200. Prove that the values of x corresponding to the maxima and minima of points on the curve $y=ae^{-Kx} \sin px$ are the values of x satisfying the equation $\tan px=\frac{p}{K}$.

201. Trace roughly the curve $y=e^{-x} \sin x$ on squared paper, and prove that the maxima and minima occur at regular equidistant intervals.

Prove that, if Y_1, Y_2, Y_3, Y_4, … are the maximum values of y, they form a series in geometrical progression.

Prove that $Y_1+Y_2+Y_3+Y_4+\ldots$ ad $\infty = \frac{e^{-\frac{\pi}{4}}}{\sqrt{2}(1-e^{-2\pi})}$, and evaluate this expression correct to two places of decimals.

202. Evaluate $\int_0^1 x\sqrt{1-x}\, dx$, $\int_0^{\frac{\pi}{2}} \frac{dx}{3+5 \cos x}$, $\int x^2 e^{3x}\, dx$. (Q.P.M.S.T.)

203. Draw the graph of $x \log_{10} x+\frac{2}{x}$ from $x=\frac{1}{2}$ to $x=8$.

Find the equation to determine the minimum value of this expression and solve this equation graphically. (Q.P.M.S.T.)

204. Obtain the complete solution of the differential equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=e^{-2t}$$

and determine the constants so that $x=0$ and $\frac{dx}{dt}=0$ when $t=0$.

From the complete solution, or in any other way, show that, when x is expressed as a power series in t,

$$x=\tfrac{1}{2}t^2-\tfrac{2}{3}t^3+\tfrac{11}{24}t^4-\ldots \qquad \text{(M.T.)}$$

205. Obtain the first few terms in the expansion of $\log_e \frac{1+x}{1-x}$ by means of a Maclaurin series. Hence calculate the value of $\log_e \frac{101}{99}$ correct to six places of decimals.

206. (i) If
$$T_n = \int_0^x \tan^n x\,dx,$$
show that
$$T_n = \frac{\tan^{n-1} x}{n-1} - T_{n-2}.$$

(ii) Given that $\pi = 3{\cdot}141592\ldots$, $\log_e 2 = 0{\cdot}693147\ldots$, show that
$$\int_0^{\frac{\pi}{4}} \tan^5 x\,dx = 0{\cdot}09657\ldots$$
$$\int_0^{\frac{\pi}{4}} \tan^4 x\,dx = 0{\cdot}11873\ldots$$
(Math. Trip.)

207. Obtain the series for e^{a+K} in ascending powers of K, and give an expression for the slope of the curve $y = e^x$ at the point where $x = a$.

A chain hanging free hangs in the form $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$. Find the equation of the tangent to the curve at the point where $x = a$, and show that the point O where $x = 0$ is the lowest point of the chain.
(Army.)

208. An anchor-ring is generated by the revolution of a circle of diameter 6 in. about a line in its own plane situated 8 in. from its centre (*i.e.* OX = 8 in.). It lies in an inch of water. Calculate the area of the surface of the anchor-ring which is wetted.

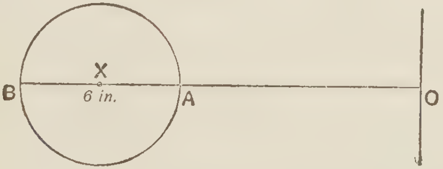

FIG. 196.

209. Evaluate (i) $\int x \log_e x\,dx$

$$\text{(ii)} \int \tan^{-1} x \, dx$$

$$\text{(iii)} \int \sec^3 x \, dx.$$

(Selwyn.)

210. Prove that $x \log_e(1+x) - x^2$ decreases as x increases from 0 to ∞. (King's.)

211. If $xy = \mathrm{A}x^2 + \mathrm{B} + \frac{1}{2}x^2 \log x$, prove that $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x$. (King's.)

212. Expand $e^{\sin x}$ in ascending powers of x as far as the term involving x^4. (Oxf. and Camb. Joint Board.)

213. Evaluate $\displaystyle\int_0^{10} \frac{x^2\,dx}{x+1}\,dx$ (first of all drawing a rough graph of $y = \dfrac{x^2}{x+1}$ in order to obtain a rough check of what to expect).

214. A motor car is driven by a constant force at all speeds, but the air-resistance increases according to the square of the velocity; show that its acceleration on the flat is given by the equation

$$\frac{dv}{dt} = k(v_0{}^2 - v^2),$$

where v_0 is its full speed. Deduce, by integrating the expression $dv/(v_0{}^2 - v^2)$, that the time taken in getting up the speed v from rest is equal to $\dfrac{1}{2kv_0} \log \left(\dfrac{v_0 + v}{v_0 - v}\right)$.

The mass of a motor car loaded is 5000 lbs., its H.P. is 50, and its full speed is 75 miles an hour. Calculate the time required to get up a speed of 50 miles an hour (starting from rest) by means of the above formula. (Army.)

215. Assume that w, the weight of a cubic centimetre of air at a height x cm. above the earth's surface, is given in terms of w_0, the weight of a c.c. at the earth's surface, by $w = w_0 e^{-\frac{x}{h}}$, where h is a constant; and that p, the pressure of a gas in kilograms-weight per sq. cm., is connected with w by the relation $p = wh$.

If, at the earth's surface, the air weighs 0·00129 gram per c.c. and the pressure there is 1·03 kilogram-weight per sq. cm., calculate the value of h.

Hence by aid of the calculus find the rate of diminution per cm. rise of the density of the air at (i) the surface, (ii) 500 metres, (iii) 1000 metres, (iv) 2000 metres above the surface. (Army Adapted.)

216. If $y^2=a^2x^2+c$, differentiate with respect to x the functions $\log_e(ax+y)$, xy, and express the results in terms of y.

Hence or otherwise evaluate the integrals $\int \frac{dx}{y}$, $\int y\,dx$. (M.T.)

217. Prove that $\int_2^{2\cdot5} \sqrt{x^2-4}\,dx=\frac{15}{8}-2\log_e 2.$ (M.T.)

218. (i) Prove that the area between the curve

$$y=ae^{-bx}\sin x$$

and the axis of x, from $x=0$ to $x=\pi$, is $\frac{a}{b^2+1}(e^{\;\;b\pi}+1)$.

(ii) If in last question $a=1$ and $b=0\cdot1$, prove that the decrease of area, when b is increased by a comparatively small quantity λ, is approximately $2\cdot6\lambda$. (Q.P.M.S.T.)

219. Change the independent variable from x to t, where $t=\log x$, in the equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$. (Peterhouse.)

220. If $y=e^{-3x}\sin 2x$, show that $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+13y=0$. (Downing.)

221. An open rectangular tank whose depth is y and base a square side x (inside measurements, in feet) is to have an inside capacity a^3 cub. ft. It is made of two pieces of metal, riveted at the four sides of the base and along one of the vertical sides. If the cost of riveting is £b per foot length of riveted seam measured inside, prove that, for the cost of riveting to be a minimum, the depth of the tank should be twice its width. (Downing.)

222. Prove that

$$\text{(i)}\quad \int_1^\infty \frac{dx}{x(x+1)^2}=\log_e 2-\tfrac{1}{2}$$

$$\text{(ii)}\quad \int_0^\infty \left\{\frac{1}{\sqrt{1+x^2}}-\frac{1}{(1+x)}\right\}dx=\log_e 2.$$ (Trinity.)

223. Find the radius of curvature at the point (x', y') on the curve

$$y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)$$

and give the co-ordinates of the centre of curvature for that point. (B.E.)

224. Find the points of inflexion on the curve $y=e^{-x^2}$. (B.E.)

225. Solve the equations:

$$\text{(i)}\ \frac{dy}{dx}=\frac{x-y}{x+y}$$

$$\text{(ii)}\ \frac{dy}{dx}+\frac{10x+8y-12}{7x+5y-9}=0. \qquad \text{(B.E.)}$$

226. Solve the equations:

$$\text{(i)}\ \frac{dy}{dx}-y=x^2$$

$$\text{(ii)}\ x(1+x^2)\frac{dy}{dx}-(1+3x^2)y=x^2-x^4. \qquad \text{(B.E.)}$$

227. Show how the equation

$$(1+x^2)\frac{dy}{dx}-xy=x^2y^2$$

may be reduced to the linear form, and hence find its solution. (B.E.)

228. Solve the following equations:

$$\text{(i)}\ \frac{d^2y}{dx^2}+7\frac{dy}{dx}+10y=e^{5x}-e^{-5x}$$

$$\text{(ii)}\ \frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=4\cos 3x$$

$$\text{(iii)}\ \frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}-\frac{dy}{dx}+y=1+x^2 \qquad \text{(B.E.)}$$

229. An electric condenser, of capacity K farads and leakage resistance R ohms, has been charged and the voltage v is diminishing according to the law

$$\frac{dv}{dt}=-\frac{v}{\mathrm{KR}}.$$

Express v in terms of the time t seconds. If $\mathrm{K}=0{\cdot}8$ microfarads (*i.e.* $0{\cdot}8\times10^{-6}$ farads); if v is noted to be 30 and 15 seconds afterwards it is noted to be 26·43, find R. (B.E.)

230. Solve $\dfrac{d^2x}{dt^2}+14{\cdot}97\dfrac{dx}{dt}+40{,}000x=0$, given that $x=0$ and $\dfrac{dx}{dt}=10$ when $t=0$. (B.E.)

231. Solve the equation $\frac{dy}{dx}=\frac{y^2-2xy}{x^2-2xy}$, and give the equation of that particular curve of the system which passes through the point $x=4$, $y=3$. (B.E.)

232. Solve $\frac{dy}{dx}+\frac{xy}{1+x^2}=\frac{1}{2x(1+x^2)}$. (B.E.)

233. Solve the equations

$$\text{(i)}\ \frac{d^2y}{dx^2}-4\frac{dy}{dx}-5y=e^{2x}+3\cos 4x$$

$$\text{(ii)}\ \frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}-6\frac{dy}{dx}=2x^3+4.$$ (B.E.)

234. In a hollow cylinder of nickel steel subjected to internal pressure p, and no pressure outside, when the material is all yielding, if p is the radial compressive stress and f the hoop tensile stress at a point whose distance from the axis is r, and if

$$f+ap=b,$$

where a and b are constants for a particular kind of steel, and if we also have the usual relation

$$\frac{dp}{dr}+p+f=0,$$

find p as a function of r. If the inside radius r_1 is 3 inches and the inside p_1 is 30 tons per square inch, what is r_0, the outer radius? [Take for nickel steel $a=\frac{3}{4}$, $b=30$.]

235. ABC is a triangle, in which the sides b and c are known exactly, but there is a small error (or uncertainty) δA in the angle A:—

(*a*) Draw AD at right angles to BC; if δa is the consequent error in the side a, show that δa is to AD in the ratio of δA to the unit of circular measure.

(*b*) If δr is the error in the radius of the inscribed circle, show that

$$a\,\delta r=r(a\cot A-r)\,\delta A.$$ (B.E.)

236. Integrate the equation

$$(1-x^2)\frac{dy}{dx}+xy=1,$$

and give the solution satisfied by the simultaneous values $x=0$, $y=1$. (B.E.)

237. Integrate the equation

$$\frac{dy}{dx}=\left(\frac{x-y-1}{2x-2y+1}\right)^2. \qquad \text{(B.E.)}$$

238. Solve completely $\frac{d^2y}{dx^2}-4a\frac{dy}{dx}+4a^2y=5\sin 2x.$ (B.E.)

239. $\frac{d^2x}{dt^2}+2f\frac{dx}{dt}+n^2x=a\sin qt$ expresses the forced vibration of a system. Imagine the natural vibrations to have been damped out. Take $n^2=49, f=3, q=5$; find x.

ANSWERS.

EXAMPLES I. (p. 13.)

1. 5 sec.　　**2.** 6 ft. per sec.　　**3.** −6 ft. per sec.

4. The graph is symmetrical about the ordinate through 5 seconds.

5. The particle moves with uniform velocity 1·5 ft. per sec.

6. The motion is uniform, the velocity being 1·5 ft. per sec.

7. The motion is uniform, the velocity being 1·5 ft. per sec.

8. The average speed is 51 miles per hr.

9. The average speed is 21 miles per hr.　　**10.** 195 ft. per sec. per sec.

11.

Time in sec.	1	2	3	4	5
Velocity in mm. per sec.	40	75	116	163	216
Acceleration in mm. per sec. per sec.			44		

12. Speed is greatest after 0·15 hour and least between 0·3 and 0·35 hour where it is zero.

13. −25 ft. per sec. per sec.

14. The gradients at $x=0$, 1, 2 are respectively 0, 3, 0.

15. 5·8 ft. per sec. per sec.

16. 24 min. after the cart passes the milestone and 2 miles beyond it.

17. 1 hour 20 min.

18. The acceleration is constant and equal to 32 ft. per sec. per sec.

19. (*a*) 1·7, (*b*) 1·4, (*c*) 1·2, (*d*) 0·7 min. per day earlier, (*e*) 0·7 min. per day later.

20. (1) 40 ft. per sec., (2) The train reaches the station B, running 5290 ft. before it stops.

EXAMPLES II. (p. 32.)

1. $4x^3$, $6x^5$, $12x^{11}$, $17x^{16}$, mx^{m-1}, $2nx^{2n-1}$, $(2n+1)x^{2n}$.

2. $6x$, $63x^8$, $112x^{13}$, $\frac{8}{3}x$, $3x^3$, $1{\cdot}5x^2$,　　**3.** 4, $8x$, $6x-12x^2$.

4. $\frac{3}{4}+\frac{8}{5}x$, $\frac{5}{2}x-\frac{7}{3}x^2$, $0{\cdot}5+1{\cdot}4x$.　　**5.** 12.　　**6.** 8.　　**7.** 12.

8. 52.　　**9.** $\frac{2}{3}$.　　**10.** 0.　　**11.** 1, $\frac{1}{3}$.　　**12.** 0·9.　　**13.** −0·4.

16. 3·62, 1·38. **17.** 75·39. **18.** 201·06. **19.** 1·25. **20.** 0·21664.

21. $\frac{dy}{dx}=16x$, 0, 1·6, 8, 32, 0°, 58°, 82° 52′, 88° 13′.

22. $\frac{dy}{dx}=8x-3$, 91° 20′, 98° 8′, 86° 55′, 87° 16′.

23. $x=0·28$. **24.** $x=0·43$. **25.** $x=0·11$ **26.** $x=0·88$, $y=-0·06$.

27. $x=-2·83$, $y=58$. **28.** $A=3x-x^2$, $x=1·5$, $A=2·25$.

29. 5, 78° 41′. **30.** 4, 75° 58′.

31.

δx	1	0·5	0·25	0·125	0·0625
$\frac{\delta y}{\delta x}$	7	4·75	3·8125	3·3906	3·1914
ϕ	81° 52′	78° 7′	75° 18′	73° 34′	72° 36′

δx	0·2	0·02	0·002	0·0002	0·00013
$\frac{\delta y}{\delta x}$	3·64	3·0604	3·006	3·0006	3·0004
ϕ	74° 38′	71° 54′	71° 36′	71° 34′	71° 34′

At $x=1$, $\frac{dy}{dx}=3$ and $\psi=71° 34′$.

32. $\frac{\delta y}{\delta x}=(0·4x+0·5)+0·2\,\delta x$. **34.** P is the middle point of AB.

35. P is the middle point of AB. **36.** 0·5.

37.

δx	0·5	0·05	0·005	0·0005	0·00005
δy	1·25	0·1025	0·010025	0·00100025	0·00010000
$\frac{\delta y}{\delta x}$	2·5	2·05	2·005	2·0005	2·00005

δx	0·3	0·03	0·003	0·0003	0·00003	$\frac{1}{4987}$
δy	0·69	0·0609	0·006009	0·00060009	0·000060001	$\frac{2}{4987}$
$\frac{\delta y}{\delta x}$	2·3	2·03	2·003	2·0003	2·00003	$2\frac{1}{4987}$

$\frac{dy}{dx}=2$. Infer that $\frac{\delta y}{\delta x}$ approaches $\frac{dy}{dx}$ in value as δx approaches 0.

38. 0·5774. **39.** 2·8 ft. for the square, 2·2 ft. for the circle.

40. $10+36x$. **41.** $-0·3-x+2·1x^2$, $-1+4·2x$. **42.** 0, $-4·8$.

43. 1·02. **44.** -31. **45.** 2. **46.** 0·7. **47.** 0·2, -5.

48. $\frac{d}{dx}\left(\frac{1}{3}x^3\right)$, $\frac{d}{dx}\left(\frac{1}{4}x^4\right)$, $\frac{d}{dx}\left(\frac{1}{5}x^5\right)$, $\frac{d}{dx}\left(\frac{5}{6}x^6\right)$, $\frac{d}{dx}\left(\frac{1}{3}x^3+x\right)$, $\frac{d}{dx}\left(\frac{2}{3}x^3-3x\right)$, $\frac{d}{dx}(x^3-x^2+4x)$, $\frac{d}{dx}\left(x^4+\frac{5}{2}x^2-9x\right)$, $\frac{d}{dx}(7x-2x^2)$.

49. $x=-3$, $x=2$, 120° 58′, 59° 2′, $y=-2$, 18° 26′.

50.

x	4	6	12	15	24
y	0·079	0·172	0·625	0·928	2
Curvature $=\frac{d^2y}{dx^2}$	0·0087	0·0078	0·0052	0·0039	0

51. 0·866.

52. A depth of 1 ft. and a width of 2 ft., the discharge then being 2 cub. ft. per sec.

53. £11 5s. **54.** £8 18s. 7d. **55.** 1185 cub. in.

56. 1·7 ft. for the triangle ; 1·3 ft. for the square.

57. x by $(20-2x)$ by $(20-2x)$ inches, Vol. $=x(20-2x)^2$ cub. in., 592·6 cub. in.

58. For a maximum $x=4·36$ in., for a minimum $x=13·31$ in.

59. 2 cub. ft. **60.** 2·55 cub. ft.

61. P is at the middle point of AB.

EXAMPLES III. (p. 44.)

1. 32, 64, 96, 128, 160 ft. per sec.

2. 11·7, 30·5, 49·3 cm. per sec. ; 9·4 cm. per sec. per sec.

3. $\frac{dT}{dt}$ degrees Centigrade per min. **4.** $\frac{dp}{dh}$ cm. per ft. rise.

5. $\frac{dp}{dT}$ lb. per sq. in. per degree Cent.

6. $-\frac{dp}{dv}$ lb. per sq. in. per cubic foot.

7. $\frac{dP}{dv}$ lb. per sq. foot per 1 mile per hour.

8 0·3 lb. wt. per cwt.

9. $2x$ sq. ft. per foot when the side of the square is x feet.

10. $3x^2$ c.c. per cm. when the side of the cube is x cm.

11. $80-32t$, 48, 16, -16, -48, -80 ft. per sec., -32 ft. per sec. per sec., after 2·5 sec., ±56·6 ft. per sec.

12. After 0·625 sec., after 3·125 sec., 156·25 feet. **13.** 93025 ft.

14. $(15-4t)$ deg. per sec., 11, 7, 3 deg. per sec., 15 deg. per sec., -4 deg. per sec. per sec., 3·75 sec.

15. 44 m. per hr. **16.** 2·6 revolutions per sec.

17. $x=0{\cdot}682y$, velocity of N $=0{\cdot}682$ velocity of P, 2·046 cm. per sec., 11·73 ft. per sec.

18. 40·42 m. per hr. **19.** 17·5 ft. per sec.

20. $y=8-2x$, W's velocity and acceleration are respectively double those of w.

21. $x=0{\cdot}105\theta$, $\dfrac{dx}{dt}=0{\cdot}105\dfrac{d\theta}{dt}$, 12·6 feet per sec.

22. $y+9x=3$. The level of the upper tank falls 9 times as fast as that of the lower rises.

23. Rate of rise of lower tank : Rate of fall of upper tank $=0{\cdot}11:1$.

24. $\mathrm{V}=15x$, $\dfrac{d\mathrm{V}}{dt}=15\dfrac{dx}{dt}$, 0·2 ft. per min.

25. 0·28 ft. per min. **26.** 1·75 cub. ft. per min.

EXAMPLES IV. (p. 57.)

1. $\dfrac{dy}{dt}=(1+2x)\dfrac{dx}{dt}$. **2.** $\dfrac{dy}{dt}=15$ sq. cm. per sec.

3. $\dfrac{dy}{dt}=(8x^3-10x+3)\dfrac{dx}{dt}$. **4.** $\dfrac{d\mathrm{V}}{dt}=3x^2\dfrac{dx}{dt}$. **5.** 1·5 c.c. per hr.

6. 314 c.c. per sec. **7.** $\dfrac{d\mathrm{V}}{dt}=(0{\cdot}9z^2-0{\cdot}2)\dfrac{dz}{dt}$, 0·2642.

8. 0·251 sq. cm. per hr. **9.** 0·02 metre per hour.

10. 51·3 miles per hour. **11.** 43·6 yards per min. **12.** 0·00071.

13. $\dfrac{dy}{dt}=-\dfrac{x}{y}\dfrac{dx}{dt}$, $-3{\cdot}75$ cm. per sec. **14.** $r=5t$ miles, 5 miles per hour.

15. 13·42 miles, $r=\sqrt{25t^2-18t+9}$ miles, $\dfrac{dr}{dt}=\dfrac{1}{r}(25t-9)$ miles per hour, 4·92 miles per hour.

16. $-2{\cdot}268$ in. per sec. **17.** 200 sq. ft. per sec. **18.** 11.

19. $-22{\cdot}2$ metres per min. **20.** 9·5. **21.** $y=6x-2$. **22.** $y=4x$.

23. Subtangent $=1{\cdot}3$. **24.** Subtangent $=\frac{1}{14}$. **25.** $\dfrac{c}{b}$.

26. Subnormal $=204$. **27.** Subnormal $=2$. **28.** Ratio $=1$.

29. $\rho=\dfrac{13\sqrt{26}}{3}$. **30.** $\rho=8$. **31.** 2·5 ft. per sec. **32.** 2·24 to 1.

33. 2·5 cub. ft., $60x^2$ cub. in., $\dfrac{dV}{dt}=120x\dfrac{dx}{dt}$ cub. in. per min.,
2·4 in. per min.

34. $V=30y^2$ cub. in., $\dfrac{dV}{dt}=60y\dfrac{dy}{dt}$ cub. in. per min., $\dfrac{dy}{dt}=3{\cdot}4$ in. per min.

35. $V=60S$ cub. in., $\dfrac{dS}{dt}=14{\cdot}4$ sq. in. per min.

36. $V=0{\cdot}1387x^3$ cub. in., $\dfrac{dV}{dt}=0{\cdot}4161x^2\dfrac{dx}{dt}$ cub. in. per sec.,
$\dfrac{dx}{dt}=166{\cdot}2$ in. per sec.

37. $S=1{\cdot}217x^2$ sq. in., $\dfrac{dS}{dt}=2{\cdot}434x\dfrac{dx}{dt}$ sq. in. per sec.,
$\dfrac{\frac{dS}{dt}}{\frac{dV}{dt}}=\dfrac{5{\cdot}849}{x}$ $\dfrac{dS}{dt}=2021$ sq. in. per sec.

38. $\dfrac{dy}{dx}=17(1+x)^{16}$. **39.** $\dfrac{dy}{dx}=30(3x-4)^9$. **40.** $\dfrac{dy}{dx}=-63(5-7x)^8$.

41. $\dfrac{dy}{dx}=10(1+x)^9\dfrac{dx}{dt}$. **42.** $\dfrac{dy}{dx}=20x(1+x^2)^9$. **43.** $\dfrac{dy}{dx}=-56x(3-4x^2)^6$.

44. $\dfrac{dy}{dx}=15(3x^2-2x)(x^3-x^2+3)^{14}$. **45.** $\dfrac{dy}{dt}=9(3-16x^3)(3x-4x^4)^8\dfrac{dx}{dt}$.

46. $\dfrac{dx}{dt}=\dfrac{1}{\pi(6x-x^2)}$ ft. per sec., 0·04 ft. per sec.

47. 0·48 cm. per min., 0·065 cm. per min.

EXAMPLES V. (p. 73.)

1. $-3x^{-4}$, $\dfrac{-5}{x^6}$, $\dfrac{-63}{x^{10}}$, $-36x^{-5}$, $-4x^{-4}$, $-5x^{-6}$, $-4x^{-2}-10x^{-3}$,
$3-5x^{-2}$, $(-n+3)x^{-n+2}$.

2. $\dfrac{3}{2}x^{\frac{1}{2}}$, $\dfrac{3}{2}x^{\frac{5}{4}}$, $3x^{-\frac{2}{5}}$, $-\dfrac{4}{3}x^{-\frac{7}{3}}$, $-x^{-\frac{8}{3}}$, $-\dfrac{7}{2}x^{-\frac{3}{2}}$, $\dfrac{1}{2}ax^{-\frac{1}{2}}-\dfrac{1}{2}bx^{-\frac{3}{2}}$,
$\dfrac{16}{3}x^{\frac{1}{3}}+\dfrac{5}{3}x^{-\frac{2}{3}}-\dfrac{14}{3}x^{-\frac{5}{3}}$. **3.** $\dfrac{1}{9}$. **4.** 0·46.

7. $-x$, $-\dfrac{4}{y}$, -1. **8.** $-\dfrac{16}{x^3}$, $-\dfrac{y^3}{4}$ -16. **9.** $-2x$, $-\dfrac{1}{2x^2}$.

11. $x(1+x^2)^{-\frac{1}{2}}$, $\frac{3}{2}x(1+x^2)^{-\frac{1}{4}}$, $-\frac{2}{3}x(1+x^2)^{-\frac{4}{3}}$, $2(3+4x)^{-\frac{1}{2}}$, $3(3+4x)^{-\frac{1}{4}}$, $-\frac{4}{3}(3+4x)^{-\frac{4}{3}}$, $\frac{3}{4}(4x-3)(2x^2-3x+7)^{-\frac{1}{4}}$, $\frac{5}{2}(4x-3)(2x^2-3x+7)^{\frac{3}{2}}$, $-5x(1-x^2)^{\frac{3}{2}}$, $-\frac{9}{2}x(1-x^2)^{\frac{5}{4}}$, $2x(1-x^2)^{-2}$, $(10+24x)(4-5x-6x^2)^{-3}$, $\frac{3}{4}(2+10x)(3-2x-5x^2)^{-\frac{7}{4}}$.

13. $\frac{2}{3}$. **15.** $-14{\cdot}04$. **17.** $y=\left(2x+\frac{8}{x}\right)$ ft., $x=2$ ft.

18. $y=\left(2x+\frac{6}{x}\right)$ ft., a rectangle 1·73 ft. deep and 3·46 ft. wide.

19. A cube of 10 ft. edge.

20. Side of the sq. $=12{\cdot}6$ ft., height of cistern $=6{\cdot}3$ ft., cost $=$ £8 18s. 7d.

21. The length of the cylinder must be 4 ft. 8 in., and radius of its circular ends 2 ft. 4 in.

22. 25 sq. in. **23.** 8·48 cm. **24.** 40 in.

25. $x=0{\cdot}5282$. **26.** $x=0{\cdot}866$.

27. 8050 cub. ft. per foot draught. **29.** 1·97 ft. per sec. per deg. Cent.

30. $\frac{2}{3}x^{\frac{3}{2}}$, $\frac{12}{7}x^{\frac{7}{4}}$, $-5x^{-1}$, $-\frac{7}{2}x^{-2}$, $-9x^{-1}-\frac{7}{2}x^{-2}$, $\frac{10}{3}x^{\frac{3}{2}}+14x^{\frac{1}{2}}$.

31. 131·3 ohms increase per 0·001 inch decrease of diameter of cross-section.

32. The sides of the triangles must be 2 ft. and length of trough 1·73 ft.

33. 0·05 sq. cm. per min. **34.** 0·25 c.c. per min.

35. $S^3=36\pi V^2$, $\frac{dV}{dt}=0{\cdot}31V^{\frac{1}{3}}$ c.c. per sec. **36.** 0·68 cm.

37. 8·66 m.p.h. **38.** 19·5 ft. per sec. **39.** 2 ft. per sec.

40. 165° 58′, 104° 2′. **41.** OT $=$ 2TN. **43.** 24 square units.

44. 1280 sq. mm. **45.** 146 mm.

46. 0·0215 gram per c.c. per min.

47. 5 ft. 6·9 in. from the 32 c.p. lamp.

48. 0·43 in. per min. **50.** $\sqrt{2x}$.

EXAMPLES VI. (p. 88.)

5. C $=3$. **6.** C $=4$. **7.** C $=7{\cdot}7$.

8. (i) $\frac{d}{dx}\left(\frac{x^3}{3}\right)$; (ii) $\frac{d}{dx}(\frac{1}{6}x^6)$; (iii) $\frac{d}{dx}(x^3)$; (iv) $\frac{d}{dx}(\frac{4}{3}x^3)$; (v) $\frac{d}{dx}(\frac{7}{6}x^6)$; (vi) $\frac{d}{dx}(\frac{1}{3}x^9)$; (vii) $\frac{d}{dx}(\frac{1}{3}x^3+x)$; (viii) $\frac{d}{dx}(x^3+\frac{1}{2}x^2)$; (ix) $\frac{d}{dx}(x^4+\frac{3}{2}x^2)$; (x) $\frac{d}{dx}(\frac{2}{3}x^3+x^2+2x)$; (xi) $\frac{d}{dx}(-x^{-1})$; (xii) $\frac{d}{dx}(-\frac{1}{2}x^{-2})$;

(xiii) $\frac{d}{dx}(-4x^{-1}+\frac{5}{2}x^{-2})$; (xiv) $\frac{d}{dx}(\frac{5}{3}x^3+4x^{-1})$; (xv) $\frac{d}{dx}\{\frac{1}{5}(x+3)^5\}$;

(xvi) $\frac{d}{dx}\{\frac{1}{16}(4x+3)^4\}$; (xvii) $\frac{d}{dx}\{(\frac{1}{25}(5x-7)^5\}$; (xviii) $\frac{d}{dx}\left\{\frac{5}{9}(x+1)^{\frac{9}{5}}\right\}$;

(xix) $\frac{d}{dx}\left\{\frac{3}{2}(x+3)^{\frac{2}{3}}\right\}$; (xx) $\frac{d}{dx}\left\{2(x-7)^{\frac{1}{2}}\right\}$; (xxi) $\frac{d}{dx}\left\{6(x-7)^{\frac{1}{3}}\right\}$.

9. (i) $\frac{x^5}{5}$; (ii) $\frac{x^{10}}{10}$; (iii) $\frac{x^{2\cdot3}}{2\cdot3}$; (iv) $\frac{x^{1\cdot2}}{1\cdot2}$; (v) $-x^{-1}$; (vi) $-\frac{1}{6}x^{-6}$;

(vii) $-\frac{x^{-1\cdot7}}{1\cdot7}$; (viii) $\frac{3}{8}x^{\frac{8}{3}}$; (ix) $\frac{2}{9}x^{\frac{9}{2}}$; (x) $5x^{\frac{1}{5}}$; (xi) $-\frac{2}{3}x^{-\frac{3}{2}}$;

(xii) $(\frac{1}{5}x^5+\frac{1}{6}x^6+\frac{1}{7}x^7)$; (xiii) $\frac{x^{2\cdot3}}{2\cdot3}+\frac{x^{3\cdot4}}{3\cdot4}+\frac{x^{4\cdot5}}{4\cdot5}$; (xiv) $(2x+\frac{3}{2}x^2+\frac{4}{3}x^3)$;

(xv) $(ax+\frac{2}{3}bx^{\frac{3}{2}})$; (xvi) $\frac{2}{1\cdot3}x^{1\cdot3}+\frac{3}{5\cdot1}x^{5\cdot1}+\frac{5}{8\cdot2}x^{8\cdot2}$; (xvii) $-\frac{1}{3x^3}$;

(xviii) $-\frac{1}{8x^8}$; (xix) $-\left(\frac{1}{x}+\frac{1}{2x^2}\right)$; (xx) $-\left(\frac{1}{6x^6}+\frac{1}{5x^5}+\frac{1}{4x^4}\right)$.

10. (i) $y=\frac{2}{3}x^9+c$; (ii) $y=x^3-\frac{5}{7}x^7+\frac{11}{8}x^8+c$;

(iii) $y=\frac{4}{3}x^{\frac{3}{2}}+\frac{3}{2}x^2-2x^{\frac{5}{2}}+c$; (iv) $11\cdot6y+x^{2\cdot32}+c=0$.

11. $y=x+\frac{1}{2}x^2-\frac{3}{2}$. **12.** $y=\frac{1}{9}x^3-\frac{5}{3}x+\frac{31}{9}$. **13.** $s=16t^2+2$.

14. $s=3\cdot1t+1\cdot15t^2$. **15.** $-\frac{11}{3}$, -5, $\frac{25}{3}$.

16. $-8\cdot5$, $-6\cdot125$, $-0\cdot765$, $6\cdot515$, $-10\cdot84$, $-10\cdot565$, $-9\cdot24$.

17. $s^{\frac{1}{2}}=4t-4+\sqrt{3}$. **18.** 28·4 ft. **19.** 43 ft.

20. $y=0\cdot0002\left(9x^2-\frac{x^3}{6}\right)$, 0·3888 inch.

21. (i) $\frac{dy}{dx}=0\cdot001(36x-x^2)$; (ii) 13° 39′, 17° 57′;

(iii) $y=0\cdot001(18x^2-\frac{1}{3}x^3)$; (iv) 1·2 in., 3·9 in.

22. (i) $\frac{dy}{dx}=\frac{1}{2}K(36x-x^2)$; (ii) $y=\frac{1}{2}K(18x^2-\frac{1}{3}x^3)$; (iii) $K=\frac{1}{648}$;

(iv) $\frac{15}{16}$ inch; (v) 10° 37′, 14° 2′.

23. $y=x^2-3x-7$. **24.** $y=\frac{1}{3}x^3+x$.

25. $v=\sqrt{0\cdot2}=0\cdot4472$ in. per sec.

26. (i) $y=\frac{1}{2}x^2$; (ii) $y=\frac{1}{2}x^2+\frac{1}{2}$; (iii) $y=\frac{1}{2}x^2-4$.

27. $y=\frac{2}{3}x^3+\frac{1}{2}x^2+c$; (i) $y=\frac{2}{3}x^3+\frac{1}{2}x^2$; (ii) $y=\frac{2}{3}x^3+\frac{1}{2}x^2-4$;

(iii) $y=\frac{2}{3}x^3+\frac{1}{2}x^2+0\cdot7$; (iv) $y=\frac{2}{3}x^3+\frac{1}{2}x^2-\frac{1}{6}$.

28. $v=\sqrt{0\cdot24}=0\cdot4899$ in. per sec. **29.** $n=0\cdot125$. **30.** $R=\frac{3}{64}\times10^5$.

31. The velocity through the mean position $=8\cdot15$ ft. per sec.

32. (i) 3·5 ft.-lb; (ii) 6·2 ft.-lb.; (iii) 7·9 ft.-lb.

33. (i) 1·25 ergs; (ii) 0·56 erg; (iii) 0·05 erg.
34. (i) 20 ergs; (ii) 4 ergs; (iii) 40 ergs.
35. 7·8275 ohms. **36.** 9·5177 ohms

EXAMPLES VII. (p. 109.)

1. 21·5 sq. units. **2.** 24·8 sq. units. **3.** 313·5 sq. ft.
4. 120 sq. in. **5.** 945 sq. units. **6.** 0·3285 sq. units.
7. 6·156 sq. in. **8.** 1·8016 sq. units. **9.** 4·0575 sq. chains.

10.

t	0·0	0·4	0·8	1·2	1·6	2·0	2·4
x	0·00	0·48	1·90	4·28	7·64	11·96	17·22
t	2·8	3·2	3·6	4·0	4·4	4·8	5·2
x	23·46	30·62	38·54	47·12	56·24	65·78	75·62

11. 36·75. **12.** 50·5. **13.** 27·75.
14. 40 ft. per sec. **15.** 211·2 units of work. **16.** 5·07 miles.
17. 418·6 joules. **18.** 11·4 ft. per sec.
19. (i) 18825 ft.-lb.; (ii) 6 ft. per sec.; (iii) 41325 ft.-lb.
20. 186·6 mils-tons.

EXAMPLES VIII. (p. 124.)

1. 108. **2.** 76·8. **3.** 1·33. **4.** 4·67 sq. in.
5. (i) 2·67; (ii) 20; (iii) 22·5; (iv) 45·33; (v) 13·34; (vi) 8·37; (vii) 5·53; (viii) 12·24; (ix) 955·9; (x) 24·65; (xi) 39·72.
6. (i) 41·67; (ii) 8·67; (iii) 573·3; (iv) 78·71.
7. (i) 24·32; (ii) 11·68; (iii) 13·2; (iv) 17·93.
8. $94\frac{1}{6}$ sq. units. **9.** 34·55 sq. units. **10.** $10\frac{2}{3}$ sq. units.
11. $166\frac{2}{3}$ sq. units. **12.** 121·5 sq. units. **13.** 288 sq. units.

14. Area OMP $= \int_0^x x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}} = \frac{2}{3}x \times x^{\frac{1}{2}} = \frac{2}{3}xy$

$= \frac{2}{3}$ rectangle OMPN.

15. Area OMP $= \int_0^x x^n\,dx = \frac{x^{n+1}}{n+1} = \frac{x \times x^n}{n+1} = \frac{xy}{n+1} = \frac{1}{n+1}$ rectangle OMPN.

18. 0·25. **19.** $\frac{n-1}{2(n+1)}$. **20.** $\frac{n-m}{(n+1)(m+1)}$. **21.** $\frac{n-1}{n+1}$.

22. 102·5. **23.** After 4 sec., 24. **24.** $\int_3^9 p\,dv$, $\int_{v_1}^{v_2} p\,dv$.

25. 166·67 cub. ft. per sec. **26.** 1·26 ft. **29.** $\frac{2}{3}$.
30. 0·565. **32.** 0·1625 ft.-lb. **33.** 8·37 ft.-lb.
34. (i) 7·5 ft.-lb. ; (ii) 13·33 ft.-lb. **35.** 60 ft.-tons.
36. 50 ergs. **37.** (i) 23·33 ergs ; (ii) 70 ergs. **38.** $\frac{500000}{C}$ ergs.
39. (i) 0·64 lb. wt. ; (ii) 0·444 lb. wt. ; (iii) $\left(\frac{4000}{4000+x}\right)^2$ lb. wt.
40. 62·75 mile-pounds.

EXAMPLES IX. (p. 137.)

1. 1·25 cub. ft. **2.** 29·25 cub. ft. **3.** 12364 cub. in.
4. 33428 cub. in. **5.** $\frac{\delta v}{\delta x}=298{\cdot}5$ sq. in. very nearly.
6. 532500 sq. ft. **7.** 532500 cub. ft. **9.** $\frac{1}{32200}$ ft. per min.
10. 330·9 c.c. **13.** 2·43 ft. **14.** 2480 cub. in.
15. 3431 cub. in. **16.** 29·3 c.c., 83·8 c.c.
17. 20·9 or 75·4 cub. ft. according as the planes are on the same or opposite sides of the centre.
18. 10·6 cub. ft. **19.** 1827·5 lb.
20. (i) 1·34 ft. ; (ii) 1·67 ft. ; (iii) 2·08 ft. **21.** 314·16.
22. 402π. **23.** 965 cub. in. **24.** 112·2 cub in.

EXAMPLES X. (p. 156.)

1. 4·47 cm. from the vertex, on the axis of symmetry.
2. On the axis of symmetry, 4·33 cm. from the shorter side.
3. Distance from AD=4·3 cm. ; from AB=1·9 cm.
4. $\bar{x}=6$. **5.** $\bar{x}=1{\cdot}43$. **6.** $\bar{y}=5$.
7. 2 inches vertically above the intersection of the diagonals of the square.
8. 5 inches vertically above the intersection of the diagonals of the rectangle.
9. 1·84 inches from each face of the tetrahedron.
10. On the axis of the cone $\frac{2}{3}h$ feet from its vertex.
11. On the pole of the tent 3 ft. $1\frac{1}{2}$ in. above the ground.
12. $\bar{x}=6{\cdot}125$ cm. **13.** $\bar{x}=\frac{3(a+b)^2}{4(2a+b)}$.
14. $\bar{x}=\frac{2}{3}b$. **15.** $\bar{x}=\frac{3a}{8}$. **16.** $\bar{y}=\frac{3b}{8}$.
17. 4·5 ft. from the pointed end.

55. $\frac{dy}{dx}=\cot\frac{\phi}{2}, \psi+\frac{\phi}{2}=\frac{\pi}{2}$. 56. $y=x^5-7x^3+4x^2-3x+9$.

57. $y=2x^5-13x^2+11x-1$. 58. $y=3(1-\sin x)$.

59. $y=5\sin x-7\cos x$. 60. $s=5t^2$.

61. $\rho=72{\cdot}76$, flex is (1, 0). 62. $\rho=8$.

EXAMPLES XIII. (p. 238.)

1. $0,\ \mp 2(1-x^2)^{-\frac{1}{2}},\ \pm 11(1-x^2)^{-\frac{1}{2}},\ \pm(9-x^2)^{-\frac{1}{2}},\ \pm\{-(2x+x^2)\}^{-\frac{1}{2}},$
$\mp 2(2-x^2)^{-\frac{1}{2}},\ \mp 2x(1-x^4)^{-\frac{1}{2}},\ \mp(49-x^2)^{-\frac{1}{2}},\ \mp 5(1-25x^2)^{-\frac{1}{2}}$.

2. $\sin^{-1}x \pm x(1-x^2)^{-\frac{1}{2}},\ 2x\cos^{-1}x \mp x^2(1-x^2)^{-\frac{1}{2}},$
$\tan^{-1}x+(1+x)(1+x^2)^{-1},\ \pm x^{-1}(1-x^2)^{-\frac{1}{2}}-x^{-2}\sin^{-1}x,$
$\mp x^{-2}(1-x^2)^{-\frac{1}{2}}-2x^{-3}\cos^{-1}x,\ \{(1+x)(1+x^2)\}^{-1}-(1+x)^{-2}\tan^{-1}x$.

3. 45°. **4.** 45°. **5.** 46° 21′, 133° 39′. **6.** 33° 52′. **7.** ±1·707.

8. ±79·16. **9.** ∓1·12. **10.** ∓146·2. **11.** 0·5737. **12.** 7·903.

13. $\mp x^{-1}(x^2-1)^{-\frac{1}{2}}$. **14.** $\pm x^{-1}(x^2-1)^{-\frac{1}{2}}$. **15.** $-(1+x^2)^{-1}$.

16. $av(x^2+a^2)^{-1}$ radians per sec. **17.** $\mp v(a^2-x^2)^{-\frac{1}{2}}$ radians per sec.

18. 1·81 degrees per sec. **19.** 8·35 degrees per sec., 4·18 degrees per sec.

20. 1·43 degrees per sec., 1·52 degrees per sec., 1·87 degrees per sec., 2·87 degrees per sec.

21. 133·6 degrees per sec. **22.** 14′ per sec., 9′ per sec., 6·2′ per sec.

23. 41·9′ per sec. **24.** 31′ per sec., 21′ per sec., 15·2′ per sec.

25. 2·05°, 0·81°, 0·42°, 0·26° per sec.

26. 0·251, 0·107, 0·598, 0·500. **27.** 0·841, 1·571, 0·745, 0·702.

28. 0·322, 0·227, 0·116, 0·076. **29.** 1·370, 0·380, 0·182, 0·456.

30. 0·322, 0·180, 0·575, 0·289. **31.** 0·416, 0·059, 0·116, 0·112.

32. 0·595, 0·155, 0·056, 0·023.

33. (i) $\theta=\alpha\sin\sqrt{\frac{g}{l}}.\,t$, (ii) $2\pi\sqrt{\frac{l}{g}}$.

34. (i) $\theta=\alpha\sin\sqrt{\frac{\mathrm{M}gl}{\mathrm{I}}}t$, (ii) $2\pi\sqrt{\frac{\mathrm{I}}{\mathrm{M}gl}}$.

35. (i) $\theta=\alpha\sin\sqrt{\frac{\mathrm{MH}}{\mathrm{I}}}t$, (ii) $2\pi\sqrt{\frac{\mathrm{I}}{\mathrm{MH}}}$.

EXAMPLES XIV. (p. 257.)

1. $\frac{\delta R}{\delta t}=15, 18, 11, 5, -4, 7$ pence per head per ann.

2. $\frac{dy}{dx}=12{\cdot}12$. **3.** $\frac{dl}{dt}=0{\cdot}217$. **4.** $\frac{dy}{dx}=1{\cdot}2$. **5.** 6·4 f.p.s.

6.

C	82	55·7	33·5	16·7	12·05	7	2·78	1·36
$\frac{dC}{dt}$	−4·56	−4·26	−2·4	−1	−1·03	−0·4	−0·14	−0·11
$\frac{dC}{dt}\div C$	−0·056	−0·076	−0·071	−0·060	−0·086	−0·057	−0·050	−0·081

$a=-0{\cdot}067$.

7. $\sqrt{x}\times\frac{dy}{dx}=0{\cdot}5$. 8. $\frac{dy}{dx}=-0{\cdot}01745 \sin x$. 9. $\frac{dy}{dx}=2{\cdot}3\times y$.

10. $\frac{dy}{dx}=0{\cdot}01745 \cot x$.

11. $\frac{\delta y}{\delta x}=0{\cdot}022$, $\delta y=0{\cdot}0066$, 0·0004, 0·0015, 0·00005.

12. $\frac{\delta h}{\delta\theta}=-2{\cdot}92$, $\delta h=-0{\cdot}0584$, $-0{\cdot}0876$, $-0{\cdot}2044$, $-0{\cdot}2638$.

13. 0·7337. **14.** $0{\cdot}01745\, p\theta \cot\theta$, $0{\cdot}01745\, p\theta \sec\theta \operatorname{cosec}\theta$.

15. +3·12. **16.** 4%. **17.** 6%. **18.** 11·5 inches.

19. 6 inches. **20.** 1·00036 times. **21.** 0·12%. **22.** $4{\cdot}8\times p$%.

23. 0·6468. **24.** $v=10$, $\delta v=0{\cdot}0333$.

25. 26, 27. $c=0{\cdot}43$, and $c=0{\cdot}4343$.

28. 0·27 cm. **29.** 0·0000103. **30.** 0·0349C cot C, 1·7%.

31. 6%. **32.** 2%. **33.** 1·1 min. **34.** 0·0134 mm.

35. 4·975 inches. **36.** 0·93 yd. **37.** 26 ft. **38.** 1·25%.

39. 11 min. **40.** 0·93%. **41.** 1%. **42.** 0·86 degrees.

43. 0·008 amp. **44.** 0·015 ohm. **45.** 6·3 yd. **46.** 2·7 ft.

47. 0·07. **48.** 0·27 sq. in. **49.** 0·0175%. **50.** 0·57%.
51. 1%. **52.** A is 0·44 degrees too large.
53. A is 0·44 degrees too small. **54.** A is 1·67 degrees too small.

EXAMPLES XV. (p. 278.)

1. (i) $\frac{x}{2}-\frac{\sin 2x}{4}$, (ii) $\frac{x}{2}-\frac{\sin 6x}{12}$, (iii) $\frac{x}{2}-\frac{\sin(10x+14)}{20}$, (iv) $-\frac{\cos^3 x}{3}$,
(v) $-\frac{\cos^4 x}{4}$, (vi) $\frac{\sin^6 x}{6}$, (vii) $\frac{\tan 3x}{3}$, (viii) $\frac{\tan 2x}{2}-x$,
(ix) $\frac{-(1-x^2)^{\frac{3}{2}}}{3}$, (x) $\frac{(1+x^2)^{\frac{3}{2}}}{3}$, (xi) $\frac{(a+bx^2)^{\frac{3}{2}}}{3b}$, (xii) $\frac{(1+x^2)^{n+1}}{2(n+1)}$,
(xiii) $\frac{(a+bx^2)^{n+1}}{2(n+1)b}$, (xiv) $\frac{\tan^4 x}{4}$, (xv) $\frac{\tan^6 x}{6}$, (xvi) $\frac{\tan^{n+1} x}{n+1}$,
(xvii) $\frac{1}{2}\sin x^2$, (xviii) $-\frac{1}{2}\cos x^2$, (xix) $\sin^{-1}(x-3)$, (xx) $\frac{1}{2}\tan^{-1} 2x$,
(xxi) $\tan^{-1}(x+1)$, (xxii) $\frac{1}{2\sqrt{3}}\tan^{-1}\left\{\frac{\sqrt{3}(x+1)}{2}\right\}$,
(xxiii) $\frac{1}{3}\sin^{-1}\left(\frac{3x}{5}\right)$, (xxiv) $\sin^{-1}\left\{\frac{x+3}{7}\right\}$, (xxv) $\frac{2}{\sqrt{3}}\tan^{-1}\left\{\frac{2x-1}{\sqrt{3}}\right\}$.

2. (i) $\frac{\pi}{4}$, (ii) 0·2727, (iii) 0·6499, (iv) 0·4961, (v) $\frac{1}{3}$,
(vi) 0·123, (vii) $\left(1-\frac{\pi}{4}\right)$, (viii) 0·2119, (ix) 0·4583, (x) 4·963,
(xi) 3, (xii) $\frac{\pi}{6}$, (xiii) $\frac{\pi}{12}$, (xiv) 0·142, (xv) $\frac{\pi}{3}$,
(xvi) $\frac{\pi}{3}$.

3. (i) $\frac{1}{2}\sin^{-1}(x^2)$, (ii) $\frac{1}{2}\tan^{-1}(x^2)$, (iii) $\frac{1}{3}\sin^{-1}\left(\frac{x^3}{3}\right)$,
(iv) $\frac{1}{15}\tan^{-1}\left(\frac{x^3}{5}\right)$, (v) $(\sin x-\frac{1}{3}\sin^3 x)$,
(vi) $(\sin x-\frac{2}{3}\sin^3 x+\frac{1}{5}\sin^5 x)$, (vii) $(-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x)$,
(viii) $\frac{2}{7}(1-x)^{\frac{7}{2}}-\frac{2}{5}(1-x)^{\frac{5}{2}}$, (ix) $\frac{2}{3}(1-x)^{\frac{3}{2}}-2(1-x)^{\frac{1}{2}}$,
(x) $\frac{2}{7}(1+x)^{\frac{7}{2}}-\frac{4}{5}(1+x)^{\frac{5}{2}}+\frac{2}{3}(1+x)^{\frac{3}{2}}$,
(xi) $\frac{1}{7}(1+x^2)^{\frac{7}{2}}-\frac{2}{5}(1+x^2)^{\frac{5}{2}}+\frac{1}{3}(1+x^2)^{\frac{3}{2}}$, (xii) $-\frac{1}{e}\sin^{-1}(e\cos\theta)$,
(xiii) $\frac{1}{e}\sin^{-1}(e\sin\theta)$, (xiv) $\cos^{-1}(x^{-1})$, (xv) $\cos^{-1}\{(ax)^{-1}\}$.

4. (i) $\frac{9}{2}\sin^{-1}\frac{x}{3}+\frac{1}{2}x\sqrt{9-x^2}$, (ii) $\frac{25}{2}\sin^{-1}\frac{x}{5}+\frac{1}{2}x\sqrt{25-x^2}$,
(iii) $\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}$, (iv) $\frac{1}{2e}\sin^{-1}(ex)+\frac{1}{2}x\sqrt{1-e^2x^2}$,
(v) $-\left[\frac{1}{2e}\sin^{-1}(e\cos\theta)+\frac{1}{2}\cos\theta\sqrt{1-e^2\cos^2\theta}\right]$,
(vi) $\frac{1}{2e}\sin^{-1}(e\sin\theta)+\frac{1}{2}\sin\theta\sqrt{1-e^2\sin^2\theta}$,
(vii) $\frac{1}{2e}\sin^{-1}(e\tan\theta)+\frac{1}{2}\tan\theta\sqrt{1-e^2\tan^2\theta}$.

5. (i) $-\left(\frac{1}{x}+\tan^{-1}x\right)$, (ii) $(\sqrt{3}\tan^{-1}x\sqrt{3}-\sqrt{2}\tan^{-1}x\sqrt{2})$,
(iii) $\frac{1}{23}\left[-\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x}{\sqrt{3}}+3\tan^{-1}3x\right]$, (iv) $-\frac{1}{x}$.

6. (i) $(\sin^{-1}x-\sqrt{1-x^2})$, (ii) $(\sin^{-1}x+\sqrt{1-x^2})$,
(iii) $-\left(\sin^{-1}\frac{1}{x}+\frac{1}{x}\sqrt{x^2-1}\right)$, (iv) $-\left(\sin^{-1}\frac{1}{x}-\frac{1}{x}\sqrt{x^2-1}\right)$,
(v) $(\sin^{-1}\sqrt{x}-\sqrt{x(1-x)})$, (vi) $2(x-1)^{\frac{1}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}$.

7. (i) $\frac{9\pi}{4}$, (ii) $\frac{49\pi}{4}$, (iii) 5·52, (iv) 2·94, (v) 17·59, (vi) 0·38.

8. (i) 0·65, (ii) 0·21, (iii) 1·33, (iv) 1·46, (v) 0·49, (vi) 0·57,
(vii) 0·48, (viii) 0·31.

9. (i) $\sin x-x\cos x$, (ii) $x\sin x+\cos x$,
(iii) $\{(3\sin x+4\cos x)+x(4\sin x-3\cos x)\}$,
(iv) $\frac{x(1+x)^{21}}{21}-\frac{(1+x)^{22}}{21\times 22}$, (v) $\frac{x(1+x)^{50}}{50}-\frac{(1+x)^{51}}{50\times 51}$,
(vi) $\frac{x(1+x)^{n+2}}{n+2}-\frac{(1+x)^{n+3}}{(n+2)(n+3)}$, (vii) $\frac{1}{4}(\sin 2x-2x\cos 2x)$,
(viii) $\frac{1}{9}(\cos 3x+3x\sin 3x)$, (ix) $x\sin^{-1}x+\sqrt{1-x^2}$,
(x) $x\cos^{-1}x-\sqrt{1-x^2}$, (xi) $\frac{1}{2}\{(1+x^2)\tan^{-1}x-x\}$,
(xii) $-\frac{1}{2}\left\{\frac{1+x^2}{x^2}\tan^{-1}x+\frac{1}{x}\right\}$, (xiii) $x^2\sin x+2x\cos x-2\sin x$,
(xiv) $-x^2\cos x+2x\sin x+2\cos x$.

10. (i) $\left\{\frac{1}{\sqrt{2}}\left(\frac{\pi}{4}+1\right)-1\right\}$, (ii) 0·2706, (iii) $\frac{\pi}{2}$, (iv) $\left(\frac{\pi}{2}-1\right)$,
(v) $\left(\frac{\pi^2}{16}+\frac{1}{4}\right)$, (vi) $\left(\frac{\pi^2}{16}-\frac{1}{4}\right)$.

11. $$\int_0^{\frac{\pi}{2}} x\sin^2 x = \int_{\frac{\pi}{2}}^{0}\left(\frac{\pi}{2}-y\right)\cos^2 y(-dy) = \int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-y\right)\cos^2 y\,dy$$
$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\cos^2 x\,dx - \int_0^{\frac{\pi}{2}} x\cos^2 x\,dx.$$

Hence $$\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\cos^2 x\,dx = \int_0^{\frac{\pi}{2}} x(\sin^2 x+\cos^2 x)dx = \int_0^{\frac{\pi}{2}} x\,dx = \frac{\pi^2}{8}$$

and thus $$\int_0^{\frac{\pi}{2}}\cos^2 x\,dx = \frac{\pi}{4}.$$

12. 1·97 sq. cm. 13. Smaller segment : larger segment $=1:100$.

14. Area $=a^2\{(\alpha-\beta)-\sin(\alpha-\beta)\,.\,\cos(\alpha+\beta)\}$ sq. cm.

15. Ratio $=(\alpha-\sin\alpha\cos\alpha):(\pi-\alpha+\sin\alpha\cos\alpha)$.

16. Ratio $=(16-9\sqrt{3}):(16+9\sqrt{3})$.

17. Ratio $=(8+\cos 3\alpha-9\cos\alpha):(8\cos 3\alpha+9\cos\alpha)$.

18. $\frac{3}{2}\pi a^2$. 19. $\frac{1}{2}a^2$. 21. $\frac{\pi a^2}{12}$. 22. $\frac{\pi a^2}{20}$. 23. πab.

24. $3\pi a^2$. 25. $\frac{3}{8}\pi a^2$. 26. $50{\cdot}5\pi$ cub. in. 27. $\frac{1}{2}\pi^2$. 28. $5\pi^2 a^3$.

29. On the axis of symmetry $\frac{4a}{3\pi}$ from the centre.

30. On the axis of symmetry $\frac{4a}{3\pi}$ from the centre.

31. On the axis of symmetry $\frac{4b}{3\pi}$ from the centre.

32. The centre of gravity is the point $\left(\frac{4a}{3\pi},\ \frac{4b}{3\pi}\right)$.

33. On the axis of symmetry of the segment and
$$\left\{\frac{2(2ah-h^2)^{\frac{3}{2}}}{3a^2\cos^{-1}\frac{a-h}{a}-3(a-h)(2ah-h^2)^{\frac{1}{2}}}-(a-h)\right\}\text{cm.}$$
from the chord of the segment.

34. On the axis of symmetry of the segment and
$$a\left\{\frac{2\sin^3\alpha}{3(\alpha-\sin\alpha\cos\alpha)}-\cos\alpha\right\}\text{in.}$$
from the chord of the segment.

35. On the axis of symmetry and $\frac{3a}{8}$ ft. from the centre.

36. On the axis of symmetry and $\left[\frac{4ah-h^2}{4(3a-h)}\right]$ cm. from the base of the cap.

37. $\frac{Ma^2}{4}$ F.P.S. units. **38.** $\frac{Mb^2}{4}$ C.G.S. units.

39. $\frac{Ma^2}{4}$ C.G.S. units. **40.** $\frac{M(a^2+b^2)}{4}$ C.G.S. units.

EXAMPLES XVI. (p. 296.)

1. (i) 20·6, (ii) 20·2, (iii) 2%. **2.** (i) 16·6, (ii) 16·9, (iii) 1·8%.

3. 12·4. **4.** 31·2. **5.** 64·2. **6.** 61·8. **7.** $\frac{(4+9a^2)^{\frac{3}{2}}-8}{27a^2}$.

8. $\frac{(9+4a^2)^{\frac{3}{2}}-8a^3}{27}$. **9.** 208·6 sq. cm. **10.** 457·7 sq. cm.

11. 186·3 sq. cm. **12.** 331·1 sq. ft. **13.** 19·87 sq. ft.

14. 10·47 sq. ft. **15.** 78·54 sq. ft. **16.** 603·3 sq. ft.

17. 220 sq. in. **18.** 1 ft. deep. **19.** (i) $6a$, (ii) $\frac{12\pi a^2}{5}$.

22. $(\pi^2-\pi)$ sq. in.

23. (i) $x=a\cos\theta$, (ii) $y=c+a\sin\theta$, (iii) $dx=-a\sin\theta\,d\theta$,
(iv) $dy=a\cos\theta\,d\theta$, (v) $(dx)^2+(dy)^2=a^2(d\theta)^2$, (vi) $ds=a\,d\theta$,
(vii) $2\times 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(c+a\sin\theta)a\,d\theta$, (viii) $4\pi^2ac$.

24. $2\pi^2a^2c$. **25.** $\frac{3\pi^2}{4}$ cub. ft., $3\pi^2$ sq. ft. **26.** (iii) $2\pi^2a^2c$.

27. Area of surface wetted $=11·61$ sq. ft.

28. £2. **29.** $2a\sqrt{2}$.

30. (i) $2\pi\left[b^2+ab\frac{\sin^{-1}e}{e}\right]$,
(ii) $2\pi\left[a^2+ab\frac{\sin^{-1}e}{e}\right]$, where $e=\sqrt{\frac{a^2-b^2}{a^2}}$.

31. (i) $a\frac{\sin\alpha}{\alpha}$, (ii) $\frac{2a}{\pi}$ distant from centre on axis of symmetry.

32. (i) $\frac{a}{2}(1+\cos a)$, (ii) $\frac{a}{2}$ distant from centre on axis of symmetry.

33. (i) $(x^4-12x^2+24)\sin x+(4x^3-24x)\cos x$,
(ii) $(5x^4-60x^2+120)\sin x-(x^5-20x^3+120x)\cos x$.

35. (i) $\frac{2}{3}$, (ii) $\frac{8}{15}$, (iii) $\frac{16}{35}$, (iv) $\frac{128}{315}$.

36. (i) $\frac{3\pi}{16}$, (ii) $\frac{5\pi}{32}$, (iii) $\frac{35\pi}{256}$, (iv) $\frac{63\pi}{512}$.

37. (i) $\frac{2}{3}$, (ii) $\frac{8}{15}$, (iii) $\frac{5\pi}{32}$, (iv) $\frac{35\pi}{256}$.

38. (i) and (ii) $\frac{(2m-1)(2m-3)\ldots 3\,.\,1}{2m(2m-2)\ldots 4\,.\,2}\times\frac{\pi}{2}$.

39. (i) and (ii) $\frac{2m(2m-2)\ldots 2\,.}{(2m+1)(2m-1)\ldots 3\,.\,1}$.

40. (i) $\frac{\pi}{4}$, (ii) $\frac{\pi}{4}-\frac{2}{3}$.

EXAMPLES XVII. (p. 309.)

1. 0·87, 0·26, 0·13, 0·05, 0·004, 0·003. 2. $-0·84$, $-0·08$, $-0·03$.

4. (i) $\frac{0·8686x}{1+x^2}$, (ii) $\frac{0·4343(1+2x)}{1+x+x^2}$, (iii) $\frac{1·3029}{3x+4}$,
(iv) $\frac{0·4343(10x-3)}{5x^2-3x-2}$, (v) $0·4343\cot x$, (vi) $-0·0076\tan x$,
(vii) $\frac{0·4343(3\cos x+4\sin x)}{(3\sin x-4\cos x)}$.

5. 0·52, 0·25, 0·43, 0·75, 0·17. 6. $\frac{0·8686}{\sin 2x}$. 7. 1·72, 1·00.

8. 1·79. 9. 1·54. 10. 3·09. 11. 0·77. 12. 0·47.

13. 1·44. 14. 0·72. 15. 0·31. 16. 0·45. 17. 0·45.

18. 0·67. 19. 0·76. 20. 0·39. 21. 0·39a. 22. 23·03.

23. 459·37. 24. 460·52, $1·157\times10^{13}$. 25. $9·149\times10^{5}$.

26. $a=\pm1·064$. 27. $a=\pm1·071$. 28. 10·67. 29. 365.

30. 0·3703. 31. 0·1389. 32. 1·5306. 33. 5·2162.

34. 12·0159. 35. 2·0138. 36. 16·939. 37. 221.

39. $a=2$ and 3. 40. 22,029. 41. 134·14. 42. 1404.

43. 836·1. **44.** 0·606.

48. 0·0986. **49.** 0·718. **50.** $e^{ax}\left(\frac{x^2}{a}-\frac{2x}{a^2}+\frac{2}{a^3}\right)$. **51.** $\frac{2}{3}$. **53.** 1.

54. $e^{\sin^2 x}\sin 2x$. **55.** $-4e^x \sin x$ **56.** $\frac{x^3}{3}\{\log_e x-\frac{1}{3}\}$.

57. $\frac{1}{12}\log_e\frac{3+2x}{3-2x}$. **58.** $\frac{7}{3}\pi$ sq. in. **59.** $x^2\log_{10}(2{\cdot}718x)=2, x=1{\cdot}726$.

60. $\sqrt{x^2-1}, \log_e\{2x+1+2\sqrt{x^2+x}\}$, 0·1277, 1. **61.** (i) $\frac{2}{\sin 2x}$,
(ii) $\frac{2x^3e^{x^2+1}}{(x^2+1)^2}$. **62.** (i) $\log_e(x+2)+\frac{1}{x+2}$, (ii) $\sin x-x\cos x$.

63. (i) $\sec x$, (ii) $(1+4x+6x^2)^{-\frac{3}{2}}$, 105° 18′. **64.** (i) $\frac{1}{2}\tan^{-1}\frac{x+1}{2}$,
(ii) $e^x(x-1)$. **65.** $e^t(\sin 3t+3\cos 3t)$. **66.** $e^x \operatorname{cosec} x(1-\cot x)$.

68. (i) 8·6507, (ii) $\frac{1}{12}$, (iii) $\frac{x^2}{4}(2\log_e x-1)$.

70. $\frac{dy}{dx}=e^{\mathrm{K}x}\{(\mathrm{KA}-\mu\mathrm{B})\sin\mu x+(\mathrm{KB}+\mu\mathrm{A})\cos\mu x\}$,
$\frac{d^2y}{dx^2}=e^{\mathrm{K}x}\{[(\mathrm{K}^2-\mu^2)\mathrm{A}-2\mathrm{K}\mu\mathrm{B}]\sin\mu x+[2\mathrm{K}\mu\mathrm{A}+(\mathrm{K}^2-\mu^2)\mathrm{B}]\cos\mu x\}$,
$\mathrm{A}=-\frac{4\mathrm{K}}{\mu}$, $\mathrm{B}=4$.

71. $\frac{2}{3}$, 0·2747, $\frac{e^{3x}}{27}(9x^2-6x+2)$.

72. $\frac{dy}{dx}=x(2\log_e x+1), \frac{d^2y}{dx^2}=2\log_e x+3$.

73. (i) 0·25, (ii) $\frac{1}{a-b}\log_e\frac{x+b}{x+a}\ (a>b)$.

74. $e^{ax}(a\sin bx+b\cos bx)$, $e^{ax}\{(a^2-b^2)\sin bx+2ab\cos bx\}$.

75. 0·1699. **76.** $\mathrm{V}=\mathrm{V}_0e^{-\frac{t}{\mathrm{KR}}}$, where V_0 is the initial voltage, 148 megohms.

77. $a=1$ and 2. **78.** $a=-2$ and -4. **79.** $a=2, n=0$ or 1.

80. $a=-3, n=0$ or 1. **81.** (i) $5cv^{0{\cdot}2}$, (ii) $c\log_e v$.

82. (i) $h=\frac{\gamma-s}{\gamma-1}p$, (ii) γ.

83. (i) $v=10e^{-2{\cdot}3t}$, (ii) $t=0{\cdot}3$ sec., (iii) $v=0{\cdot}1$ cm. per sec.,
(iv) $v=10-2{\cdot}3s$, (v) 1·3 cm., (vi) $s=4{\cdot}35(1-e^{-2{\cdot}3t})$,
(vii) 4·344 cm.

84. (i) $K=0{\cdot}2$, (ii) 27 cm., 6·6 cm. per sec., (iii) 0·08 sec.

85. 5633 metres. **86.** $8{\cdot}851 \times 10^5$ dynes per sq. cm.

87. 23 min. 51 sec. **88.** $T=24{\cdot}6$.

89. $T=\theta-\dfrac{Q}{2\pi kl}\log_e r$. **90.** 12°. **91.** 2·9 cm. **92.** 26·4.

93. 816·2 joules. **94.** $\dfrac{2Q}{K}\log_e\dfrac{r_2}{r_1}$, $\dfrac{K}{2\log_e\dfrac{r_2}{r_1}}$.

95. 148,500 electrostatic units. **96.** $T=T_0e^{\mu\theta}$. **97.** 35 lbs. wt.

98. 48·75 cwt. **99.** 0·214. **100.** 4·7 turns.

EXAMPLES XVIII. (p. 337.)

1. (i) $y=0{\cdot}5x+0{\cdot}3424$, (ii)

x	60° 10′	60° 23′	60° 35′	60° 48′	61°
Calculated value of $\sin x$	0·8675	0·8694	0·8711	0·8730	0·8747
Value of $\sin x$ from the Tables	0·8675	0·8694	0·8711	0·8729	0·8746
Number of places of decimals to which you are correct	4	4	4	3	3

2. (i) $y=1{\cdot}4068-0{\cdot}8660x$, (ii)

x	60° 17′	60° 38′	61°	62°	65°	70°
Calculated value of $\cos x$	0·4957	0·4904	0·4849	0·4698	0·4244	0 3489
Value of $\cos x$ from the Tables	0·4957	0·4904	0·4848	0·4695	0·4226	0·3420
Number of places of decimals to which you are correct	4	4	3	3	2	2

3. (i) $y=1{\cdot}333x-0{\cdot}1208$, (ii)

x	30° 5′	30° 18′	30° 37′	31°	32°	33°
Calculated value of tan x	0·5793	0·5843	0·5916	0·6006	0·6238	0·6471
Value of tan x from the Tables	0·5794	0·5844	0·5918	0·6009	0·6249	0·6494
Number of places of decimals to which you are correct	3	3	3	3	2	2

4. (i) $y=x$, (ii) 0·0300. 5. (i) $y=x$, (ii) 0·004.
6. 0·5256 radians. 7. 5·0483. 8. 6·1510.
9. (i) $y=1{\cdot}1198-1{\cdot}7320x$, (ii) $-0{\cdot}7119$. 10. $-0{\cdot}1348$.
11. 0·0157. 12. 1·0017. 13. 6·0972. 14. $y=3x$. 15. $y=0$.

16. $\left(y-\frac{\sqrt{3}}{2}\right)=\frac{1}{2}\left(x-\frac{\pi}{3}\right)-\frac{\sqrt{3}}{4}\left(x-\frac{\pi}{3}\right)^2$,

sin 60° 37′ $=0{\cdot}8713569$ to six places correct,
sin 63° $=0{\cdot}8910180$ to four places correct.

17. $y-1=-2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2$,

cot 45° 33′ $=0{\cdot}9809857$ correct to five places,
cot 50° $=0{\cdot}8406978$ correct to two places.

18. $y=1+x+\frac{x^2}{2}+\frac{x^3}{6}$, $e^{0\cdot03}=1{\cdot}0304545$, $e^{-0\cdot04}=0{\cdot}9607893$.

19. $y=x-\frac{x^2}{2}+\frac{x^3}{3}$, $\log_e 1{\cdot}03=0{\cdot}029559$, $\log_e 0{\cdot}996=-0{\cdot}004008$.

20. 2·0167050.

21. $y=1+\frac{3}{4}x-\frac{3}{32}x^2+\frac{5}{128}x^3$, $(1{\cdot}05)^{\frac{3}{4}}=1{\cdot}0372705$, $(0{\cdot}998)^{\frac{3}{4}}=0{\cdot}9984996$.

22. $y=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right)-\frac{1}{2\sqrt{2}}\left(x-\frac{\pi}{4}\right)^2$. 23. $y=x+\frac{1}{6}x^3$.

24. $\sin\left(\frac{\pi}{4}+x\right)=\frac{1}{\sqrt{2}}+\frac{x}{\sqrt{2}}-\frac{x^2}{2\sqrt{2}}-\frac{x^3}{6\sqrt{2}}+\frac{x^4}{24\sqrt{2}}+\dots$

The first four terms must be included to give a result correct to five places of decimals for all angles $\not>$ 2°.

25. $f(x)=f(0)+xf(0)+\frac{x^2}{\underline{|2}}f''(0)+\ldots+\frac{x^r}{\underline{|r}}f^r(0)+\ldots$

$e^x=1+x+\frac{x^2}{\underline{|2}}+\ldots+\frac{x^r}{\underline{|r}}+\ldots$

$e=2{\cdot}718254.$

27. $0{\cdot}893.$

28. $\log_e 1{\cdot}01=0{\cdot}0099503$, $\log_{10} e=0{\cdot}43427.$

29. $f(a+x)=f(a)+xf'(a)+\frac{x^2}{\underline{|2}}f''(a)+\frac{x^3}{\underline{|3}}f'''(a)+\ldots$

$\sin(a+x)=\sin a+x\cos a-\frac{x^2}{\underline{|2}}\sin a-\frac{x^3}{\underline{|3}}\cos a+\ldots$

$\sin 31°=0{\cdot}5150.$

30. $\frac{1}{\sqrt{1+x^2}}=1-\frac{1}{2}x^2+\frac{3}{8}x^4-\frac{5}{16}x^6$, $\frac{1}{\sqrt{10}}=0{\cdot}3162.$

31. $0{\cdot}03343.$

32. $0{\cdot}66$ per cent.

33. $\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}.$

34. $0{\cdot}000047936.$

35. $\log_e(1+\sin x)=x-\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}-\frac{2x^4}{\underline{|4}}.$

37. The re-written equation is $y=\frac{x^2}{8}.$

38. $e^{\tan x}=1+x+\frac{x^2}{\underline{|2}}+\frac{3x^3}{\underline{|3}}+\frac{9x^4}{\underline{|4}}+\frac{37x^5}{\underline{|5}}.$

EXAMPLES XIX. (p. 349.)

1. $\frac{1-x^{n+1}}{(1-x)^2}-\frac{(n+1)x^n}{(1-x)}=\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}.$

2. $[2-n(n+1)x^{n-1}+2(n^2-1)x^n-n(n-1)x^{n+1}](1-x)^{-3}.$

3. $[1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}](1-x)^{-3}.$

4. $6(1-x)^{-4}.$

5. $y=(1-x)^{-\frac{1}{2}}.$

6. Write x^2 for x in the result of 5.

7. $\sin^{-1}x=x+\frac{1}{2}\frac{x^3}{3}+\frac{1\,.\,3}{2\,.\,4}\frac{x^5}{5}+\ldots+\frac{1\,.\,3\,.\,5\ldots(2n-1)}{2\,.\,4\,.\,6\ldots 2n}\frac{x^{2n+1}}{2n+1}+\ldots \text{ad } \infty.$

8. $y=(1+x)^n.$

9. $\frac{dy}{dx}=y$, $y=e^x.$

10. $\log_e\sqrt{1+x^2}=\frac{x^2}{2}-\frac{x^4}{4}+\frac{x^6}{6}-\frac{x^8}{8}+\ldots+(-1)^{n-1}\frac{x^{2n}}{2n}+\ldots \text{ad } \infty.$

11. $\log \sqrt[3]{1+x^3} = \frac{x^3}{3} - \frac{x^6}{6} + \frac{x^9}{9} - \frac{x^{12}}{12} + \ldots + (-1)^{n-1}\frac{x^{3n}}{3n} + \ldots$ ad ∞.

The expansion is valid for $+1 > x > -1$.

12. $\log_e \sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2n+1}}{2n+1} + \ldots$ ad ∞.

13. $\frac{1}{x}\frac{dy}{dx} = -(1-x^2)^{-\frac{1}{2}}$, $y = \sqrt{1-x^2}$.

14. $y = \sin x$, because $y = 0$ when $x = 0$.

15. $y = \cos x$, because $y = 1$ when $x = 0$.

EXAMPLES XX. (p. 378.)

1. $y = 3x + \mathrm{A}$. 2. $x^2 + y^2 = c^2$. 3. $y^2 + 6x = \mathrm{A}$.

4. $xy = c^2$. 5. 18° 26′. 6. $x^2 + y^2 = 1$.

7. $x - 2y + 3 = 0$. 8. $3x - 4y + 5 = 0$.

9. (i) $\frac{x+y}{1-xy} = c$, (ii) $\cos x \cos y = c$, (iii) $(1+x)^3 + 2(1-y)^4 = c$,

(iv) $3(1-y)^{-2} + (1+x)^{-6} = c$, (v) $\sin^{-1} y + e^{-x} = c$.

10. (i) $y = 1 + c\,e^{-\frac{1}{2}x^2}$, (ii) $y = (x^2 - 2) + c\,e^{-\frac{1}{2}x^2}$, (iii) $y = 1 + c\,e^{\frac{1}{x}}$,

(iv) $y = \sin x - 1 + c\,e^{-\sin x}$, (v) $y = \cos x - 1 + c\,e^{-\cos x}$.

11. $y = \left(\frac{x+2}{x+1}\right)^2 \left\{x + \frac{1}{x+2} + c\right\}$.

12. $y = \cos x + c \cos^2 x$.

If $y = 0$ when $x = \frac{\pi}{3}$, $c = -2$, and for a maximum y we have $\frac{dy}{dx} = 0$,

hence $y = \cos x - 2\cos^2 x$, and $2y \tan x = \sin x$ or $y = \frac{1}{2}\cos x$.

Thus $8y^2 - y = 0$, giving $y = \frac{1}{8}$.

13. $x = \frac{1}{\mu}\left(t - \frac{1}{\mu}\right) + \frac{1}{\mu^2}e^{-\mu t}$, ∞. 14. $y = \frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}(1 - e^{-ax})$, ∞.

15. $\mathrm{C} = \frac{\mathrm{E}}{\mathrm{R}}\left(1 - e^{-\frac{\mathrm{R}t}{\mathrm{L}}}\right)$, $\mathrm{C} = \frac{\mathrm{E}}{\mathrm{R}}$. 16. $y = x^3 + \mathrm{K}x^{-1}$.

17. $y = \tan x + \mathrm{K} \sec x$. 18. $y = \sin x + \mathrm{K} \cos x$.

19. $y = \cos x + \mathrm{K} \sin x$. 20. $y = e^x(\sin x + \mathrm{K})$.

21. $y=\sin^2 x+\mathrm{K}\sin x$. 22. $x^2(x-y)=\mathrm{K}$.

23. $y=\mathrm{K}e^{-\frac{x}{y}}$. 24. $\log_e(x^2+y^2)=2\tan^{-1}\frac{y}{x}+\mathrm{K}$.

25. (i) $(x-y)e^{\frac{x}{x-y}}=\mathrm{K}$, (ii) $x^2+y^2=\mathrm{K}x^4$.

26. General solution, $y=cx-5c^3$; singular solution, $135y^2=4x^3$.

27. General solution, $y=cx+(1+c^2)$; singular solution, $4y+x^2=4$.

28. General solution, $y=cx+c+c^2$; singular solution, $4y+(x+1)^2=0$.

29. General solution, $y=cx+3c^2$; singular solution, $12y+x^2=0$.

30. General solution, $y=cx-\frac{4}{c}$; singular solution, $y^2+16x=0$.

31. General solution, (i) $y=cx+\sqrt{1+c^2}$, (ii) $y=cx+c^{-1}$,
(iii) $y=cx+(c-c^2)$, (iv) $y=cx-(2c+c^3)$,
(v) $y=cx+c(1+c^2)^{-\frac{1}{2}}$;

Singular solution, (i) $x^2+y^2=1$, (ii) $y^2=4x$,
(iii) $4y=(x+1)^2$, (iv) $27y^2=4(x-2)^3$,
(v) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$.

33. When $x<3a$, y is +ve, $\frac{d^2y}{dx^2}$ is −ve,

" $x=3a$, $y=0$, $\frac{d^2y}{dx^2}=\infty$,

" $x>3a$, y is −ve, $\frac{d^2y}{dx^2}$ is +ve,

and these facts give the required results.

35. $3ay^2=2x^3$. 36. A straight line. 37. A straight line.

38. Its equation is $y^2=k^2x^2+l^2$, and the curve is a conic or a pair of straight lines.

39. $x+\sqrt{1-y^2}=\log_e(1+\sqrt{1-y^2})+(1-\log_e 2)$.

40. The straight line $y=x$. 41. $2y=x^2$. 42. $y=ke^{\frac{x}{a}}$.

43. $y+\log_e\cos x=0$. 44. $y=ke^x+l$. 45. $\mathrm{V}=-\frac{k}{r}+l$.

46. $\left(\frac{du}{d\theta}\right)^2+u^2=\frac{2\mu}{h^2}u+k$. 47. $y=(\mathrm{A}x+\mathrm{B})e^{2x}$.

48. Velocity at earth's surface $=\sqrt{2ga}$. Distance after time $t=\left\{d^{\frac{3}{2}}-\left(\frac{9}{2}ga^2\right)^{\frac{1}{2}}t\right\}^{\frac{2}{3}}$, where d is the distance from the earth's centre of the starting particle.

49. $v = v_0 e^{-\mu t}$, $x = a + \frac{v_0}{\mu}(1 - e^{-\mu t})$, $v = v_0 + \mu(a - x)$.

After a very long time particle comes to rest at distance $= a + \frac{v_0}{\mu}$.

50. (i) $y = 1 + 2x + 3x^2 + \frac{14}{3}x^3$, (ii) $1{\cdot}0628$.

51. $$y = 1 + \frac{2}{\pi}\left(x - \frac{\pi}{2}\right) - \frac{\pi}{4}\left(x - \frac{\pi}{2}\right)^2 - \frac{1}{2}\left(x - 2\right)^3 + \frac{\pi}{48}\left(x - \frac{\pi}{2}\right)^4.$$

52. (i) $y = \frac{5}{4}x - \frac{9}{32}x^2 + \frac{3}{16}x^3$, (ii) $y = 0{\cdot}04956$,
(iii) $y = 1 + (x - 1) - \frac{1}{14}(x - 1)^2 + \frac{4}{147}(x - 1)^3$.

53. $\frac{d\theta}{dt} = \pm\sqrt{\frac{2g}{a}(\cos\theta - \cos\alpha)}$.

54. $v = \mathrm{V}e^{-\mu(x-a)}$, $x = a + \frac{1}{\mu}\log_e(1 + \mu \mathrm{V}t)$.

55. $\mathrm{K}x = \frac{1}{m-1}\left(\frac{1}{v^{m-1}} - \frac{1}{u^{m-1}}\right)$, $\mathrm{K} = 10^{-2}$, 2000 ft.

56. $\dot\theta = \left[2\left(\frac{g}{a} - \frac{\lambda}{ml}\right)(\cos\theta - \cos\alpha) + 4\frac{\lambda}{am}\left(\cos\frac{\theta}{2} - \cos\frac{\alpha}{2}\right)\right]^{\frac{1}{2}}$.

57. $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \mu^2 xy = 0$. **59.** $\left[(c + x)\frac{dx}{dt}\right]^2 = \mathrm{K} - \frac{2}{3}g(c + x)^3$.

60. $x = a + \mathrm{A}\cos ct + \mathrm{B}\sin ct$.

61. $x = \mathrm{A}t + \mathrm{B}$, $y = \sqrt{\mathrm{C}t^2 + 2\mathrm{D}t + \frac{\mathrm{D}^2 - 1}{\mathrm{C}}}$.

62. $r = \mathrm{A}e^{\omega t} + \mathrm{B}e^{-\omega t} + \frac{g}{2\omega^2}\sin\omega t$. **63.** $x = \mathrm{A}e^t + \mathrm{B}e^{-t}$ $y = \mathrm{A}e^t - \mathrm{B}e^{-t}$.

64. $z = (\mathrm{A}t + \mathrm{B})e^{\frac{3t}{2}}$. **65.** $y = \mathrm{A}e^x + \mathrm{B}e^{-6x}$. **66.** $y = \mathrm{A}e^{\frac{1}{2}x} + \mathrm{B}e^{\frac{5}{4}x}$.

67. $y = \mathrm{A}e^{3x} + \mathrm{B}e^{-2x} - \frac{1}{3}x + \frac{1}{18}$. **68.** $y = \mathrm{A}e^{3x} + \mathrm{B}e^{4x} + \frac{1}{4}x + \frac{23}{48}$.

69. $y = \mathrm{A}e^{5x} + \mathrm{B}e^{-x} - \frac{1}{5}x^2 + \frac{4}{75}x - \frac{38}{1125}$.

70. $y = \mathrm{A}e^{4x} + \mathrm{B} + \frac{3}{20}(2\cos 2x - \sin 2x)$.

71. $y = \mathrm{A}e^t + \mathrm{B}e^{2t} + \frac{1}{10}(\sin t + 3\cos t)$. **72.** $y = \mathrm{A}\cos 2t + \mathrm{B}\sin 2t + \frac{1}{13}e^{3t}$

73. $y = \mathrm{A}e^{3t} + \mathrm{B}e^{-3t} - \frac{1}{4}e^t + \frac{3}{5}e^{2t}$.

74. $y = e^{-t}(\mathrm{A}\cos 2t + \mathrm{B}\sin 2t) + \frac{1}{16}(2t - 1)e^t$.

MISCELLANEOUS EXAMPLES III. (p. 386)

CHAPS. XI.–XV.

107. Volume $=\dfrac{25\pi^2}{8}$, $\xi=\dfrac{\pi}{8}+\dfrac{1}{2\pi}$.

108. $f(x)\equiv\dfrac{5x^2-2x+5}{2x^2+2}$. **112.** $\dfrac{1}{p}\left\{\sin\dfrac{p\pi}{2}-\cos\dfrac{p\pi}{2}+1\right\}$.

113. $2-\dfrac{\pi}{2}=0{\cdot}4292$. **114.** 0·946 sec., $-12{\cdot}648$.

118. 17·7 yds.

119. (i) $\dfrac{3(x-1)}{(2x^2-2x+5)^{\frac{3}{2}}}$, (ii) $\dfrac{2x-3x^2-x^4}{(1+x^3)^2}\cos\dfrac{1+x^2}{1+x^3}$.

121. Condition that function always increases is that $(ae-bc)$ is positive.

122. About 0·07%.

124. (i) $-e^{-x}\{(\mathrm{A}-\mathrm{B})\cos x+(\mathrm{A}+\mathrm{B})\sin x\}$, (ii) $\frac{1}{2}(1+e^{-\frac{\pi}{2}})$.

126. $\left(\dfrac{\pi}{6}-\dfrac{2}{9}\right)$. **127.** 0·5.

131. (i) $\dfrac{x^2}{4}-\dfrac{x\sin 2x}{4}-\dfrac{\cos 2x}{8}$, (ii) $\dfrac{\pi}{3\sqrt{3}}$, (iii) $\dfrac{1}{3\sqrt{2}}$.

132. The maximum value is $-\dfrac{\sqrt{3}}{2}$ when $x=-2+\sqrt{3}$.

The minimum value is $+\dfrac{\sqrt{3}}{2}$ when $x=-2-\sqrt{3}$.

134. 0·7%. **136.** $\frac{2}{15}$.

137. Volume $=\pi a^3\left\{2\sin\dfrac{\alpha}{2}-\frac{2}{3}\sin^3\dfrac{\alpha}{2}-\alpha\cos\dfrac{\alpha}{2}\right\}$. **139.** 3·5%.

140. (i) $\dfrac{\pi^2}{18}$, (ii) $\frac{2}{3}\pi\left(\dfrac{\pi}{2}-1\right)$. **141.** $2x^2-8x-4y+5=0$.

144. $\delta p=\dfrac{a}{\sin^2(\theta+\phi)}\{\sin^2\phi\,\delta\theta+\sin^2\theta\,\delta\phi\}$.

145. $x^3\sin x+3x^2\cos x-6x\sin x-6\cos x$. **146.** $\frac{1}{2}\pi a^2 b$.

149. $y+4m^3x-6am^2=0$.

150. Area $=71{\cdot}37$ sq. in., percentage errors are 0·275 and 0·0385.

152. $\mathrm{A}=\frac{518}{225}a$, $\mathrm{V}=\frac{259}{450}\pi^2a^3$. **154.** (i) 250 sq. ft., (ii) About 44′.

155. $\left\{\dfrac{\sin(m-n)\frac{\pi}{2}}{2(m-n)} - \dfrac{\sin(m+n)\frac{\pi}{2}}{2(m+n)}\right\}$. 156. $\left\{\dfrac{\pi}{4} - \dfrac{\sin m\pi}{4m}\right\}$.

157. 2 bc units.

MISCELLANEOUS EXAMPLES IV. (p. 391.)

CHAPS. XVI.–XX.

161. $\dfrac{dS}{dx} = 4\sqrt{11} \,.\, \pi$.

163. (i) $\dfrac{x^2}{4} - \dfrac{x\sin 2x}{4} - \dfrac{\cos 2x}{8}$, (ii) $-2x(1-x)^{\frac{1}{2}} - \frac{4}{3}(1-x)^{\frac{3}{2}}$, (iii) $\dfrac{\pi}{2}$.

164. Length of loop $= \dfrac{4a}{\sqrt{3}}$, area of loop $= \dfrac{8a^2}{15\sqrt{3}}$.

165. $\log_e \frac{32}{27} = 0{\cdot}1699$. 167. $\cot 5^\circ = 11{\cdot}4300$.

169. $\log_e \left\{\dfrac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right\}$, $\tan\dfrac{x}{2}$.

170. Length of one quadrant $= \dfrac{a^2+ab+b^2}{a+b}$.

171. $\sin\left(\dfrac{\pi}{4}+\theta\right) = \dfrac{1}{\sqrt{2}}\left\{1+\theta-\dfrac{\theta^2}{\underline{|2}}-\dfrac{\theta^3}{\underline{|3}}+\dfrac{\theta^4}{\underline{4}}+\dfrac{\theta^5}{\underline{|5}}-\ldots\right\}$.

172. (i) $\sec x$, (ii) $(1+4x+6x^2)^{-\frac{3}{2}}$, $-5e^{-\frac{\pi}{10}}$.

173. $\dfrac{1}{4}\log_e\dfrac{x-1}{x+1} - \dfrac{1}{2} \cdot \dfrac{1}{(x+1)}$.

174. Length of curve $= 8a$, area enclosed by curve $= \frac{3}{2}\pi a^2$.

175. $\dfrac{1}{6}\left\{\dfrac{10}{x-1} - \dfrac{9}{x} - \dfrac{1}{x+2}\right\}$, $\log_e\left\{(x-1)^{\frac{5}{3}} x^{-\frac{3}{2}}(x+2)^{-\frac{1}{6}}\right\}$.

176. (i) $8{\cdot}6507$, (ii) $\frac{1}{12}$, (iii) $\dfrac{x^2}{2}(\log_e x - \frac{1}{2})$.

177. 49° 31′, 164° 33′, 43·43. 178. $\frac{19}{27}a$. 179. (i) $\dfrac{\pi}{4}$

180. x^2. 181. 10·75. 182. $2x - 3\log_e \frac{1}{2}(x+2)$.

233. (i) $y = Ae^{5x} + Be^{-x} - \frac{1}{9}e^{2x} - \frac{1}{697}(63 \cos 4x + 48 \sin 4x)$,
(ii) $y = Ae^{3x} + Be^{-2x} - \frac{1}{12}x^4 + \frac{1}{18}x^3 - \frac{7}{36}x^2 - \frac{59}{108}x + C$.

234. $p = Ar^{a-1} + \dfrac{b}{a-1}$, $r_0 = 7{\cdot}3$ in.

236. $y = x + C(1 - x^2)^{\frac{1}{2}}$, $y = x + \sqrt{1 - x^2}$.

237. $\log_e \dfrac{(x - y + 2)^9}{(x - y)} - 2x + 8y = C$.

238. $y = (A + Bx)e^{2ax} + \dfrac{5}{4(a^2 + 1)^2}\{(a^2 - 1) \sin 2x + 2a \cos 2x\}$.

239. $x = \dfrac{a}{246}(4 \sin 5t - 5 \cos 5t)$.

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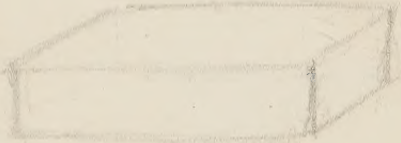
$$\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\frac{\cancel{(s-}b)(s-c)}{bc} \times \frac{\cancel{s(s-c)}}{s(s-a)}}$$

Gradient

30 Forrest Place



Obv.

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